Toda conjecture mod $p > 3$

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Toda constructed and used (see 'On exact sequences in Steenrod algebra mod 2', Memoirs of the College of Science, University of Kyoto. Series A: Mathematics 31 (1958), no. 1, 33–64) some exact sequences for quotients of the Steenrod algebra $\mathcal{A}_2$. He also conjectured the exactness of the sequence

$$\mathcal{A}_2 \xrightarrow{\varphi_r} \mathcal{A}_2/\mathcal{A}_2\mathcal{S}_{r-2} \xrightarrow{\varphi_r} \mathcal{A}_2/\mathcal{A}_2\mathcal{S}_{r-1} \tag{1}$$

where $\varphi_r(x) = x \cdot Sq^{2^r}$ and $\mathcal{S}_k$ is the subalgebra generated by the elements $Sq^1,\ Sq^2,\ Sq^4, \dots, Sq^{2^k}$. The conjecture was proved by Wall in 'Generators and relations for the Steenrod algebra', Ann. of Math. (2) 72 (1960), 429444.

Consider prime $p > 3$. Let a and b be integers such that $a, b \in [1, p-1]$ and $a + b = p$. Define $\alpha_i : \mathcal{A}_p \xrightarrow{\mathcal{A}_p} /\mathcal{A}_p\mathcal{S}_{r-2}$ by $\alpha_1(x) = x \cdot P^{ap^r}$ and $\alpha_2(x) = x \cdot (P^{p^r})^a$ and also $\beta_i : \mathcal{A}_p/\mathcal{A}_p\mathcal{S}_{r-2} \xrightarrow{\mathcal{A}_p} /\mathcal{A}_p\mathcal{S}_{r-1}$ by $\beta_1(x) = x \cdot P^{bp^r}$ and $\beta_2(x) = x \cdot (P^{p^r})^b$.

One can show that for any $a + b = p$ and $i, j \in \{1, 2\}$ there are well defined homomorphisms of quotients

$$\mathcal{A}_p \xrightarrow{\alpha_i} \mathcal{A}_p/\mathcal{A}_p\mathcal{S}_{r-2} \to \beta_j \mathcal{A}_p/\mathcal{A}_p\mathcal{S}_{r-1}. \tag{2}$$

Moreover one has $\beta_j \circ \alpha_i = 0$.

Exactness of the sequence (2) is the natural generalization of the Toda conjecture for the case $p > 2$.

Theorem. (a) The element $2Z_{r-1}^{r} Z_{r-2}^{r-1} - Z_{r-2}^{r} Z_{r-1}^{r-1}$ belongs to the kernel β_j, but does not belong to the image of α_i.

(b) For $\beta(x) = x \cdot P^{(p-1)p^r}$ and $\alpha(x) = x \cdot P^{p^r}$ the sequence is exact in lower grading.

(c) In grading $2(p-1)k = 2(p-1)(2p^{r-1} + p^{r-2})$ the quotient $ker\ \beta / im\ \alpha$ is of dimension one.

The proof is based on the results from the paper 'On monomial bases in the mod(p) Steenrod algebra' by D. Yu. Emelyanov, Th. Yu. Popelensky, Journal of Fixed Point Theory and its Applications. 2015, Vol. 17, Num. 2, pp. 341-353

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