Top dense ball packings and coverings in hyperbolic space

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In the classical Euclidean 3−space the so-called Kepler conjecture on the densest packing $\mathbb{E}^3$ with congruent balls (with density $0.74\ldots$) has been recently solved by Thomas Hales by computer, following the strategy of László Fejes Tóth (1953). In the Bolyai-Lobachevsky hyperbolic space we know only a density upper bound (K. Böröczky and A. Florian (1964)), realized (only) by horoballs in ideal regular simplex arrangement with density $0.85\ldots$, and the realization is not unique (R.T. Kozma and J. Szirmai [2]). With proper balls we are far from this packing upper bound, and there is no real chance yet for the more difficult ball covering problem in $\mathbb{H}^3$.

Our aim in this work is a systematic computer experiment to attacking both problems for packing and covering by a construction scheme. These ball arrangements will be based on complete (or extended) Coxeter orthoscheme groups, generated by plane reflections.

E.g. the Coxeter-Schläfli symbol $(u, v, w) = (5, 3, 5)$ describes first the characteristic orthoscheme of a regular dodecahedron (as $(5, 3, \,.\,)$) refers to it) with dihedral face angle $2\pi/5$ (as $(\,.\,,\,.\,, 5)$ indicates it). This dodecahedron - by its congruent copies - fills $\mathbb{H}^3$ just by the reflections in the side faces of the above orthoscheme (characterized also by a Coxeter-Schläfli matrix, scalar product of signature $(+ + +-)$, etc.). This orthoscheme $A_0A_1A_2A_3$ has also a half-turn symmetry $0 \leftrightarrow 3$, $1 \leftrightarrow 2$ that extends to the complete symmetry group of the $\mathbb{H}^3$ tiling. Not surprisingly, we get the H3 tiling with the hyperbolic football (the Archimedean solid $\{5, 6, 6\}$) as in the earlier works [3, 4] of the first author. The central ball (centred in A_3 or in A_0) in the above football solid has a packing density $0.77\ldots$, as a top density so far, just discovered now.

For this and the analogous generalized further series, a volume formula of orthoscheme by N.I. Lobachevsky (1837) was needed, that has been extended to complete (or truncated) orthoschemes by R. Kellerhals [1]. The second author intensively worked on its computer program (see e.g. [5]). Thus we get a large list of good (top!?) constructions as for packing densities as for covering ones as well, together with their metric data. For these the ball centre also varies on the surface of the (truncated) orthoscheme, together with the ball radius. So we have to implement large computations, indeed.

This is a joint work with Jenő Szirmai.

[1] Kellerhals R. On the volume of hyperbolic polyhedra. Mathematische Annalen. 1989; 245: 541-569.

[2] Kozma RT, Szirmai J. Optimally dense packings for fully asymptotic Coxeter tilings by horoballs of different types. Monatshefte für Mathematik. 2012; 168(1): 27-47.

[3] Molnár E. Two hyperbolic football manifolds. Proceedings of International Conference on Differential Geometry and Its Applications, Dubrovnik Yugoslavia. 1988; 217-241.

[4] Molnár E. Tnon-Euclidean crystallography, some football manifolds. Journal of Structural Chemistry. 2012; 23: 1057-1069.

[5] Szirmai J. The optimal hyperball packings related to the smallest compact arithmetic 5-orbifolds. Kragujevac Journal of Mathematics, to appear.