Integrable complex structures on nilpotent Lie algebras

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An almost complex structure J on a Lie algebra $\mathfrak{g}$ ($J : \mathfrak{g} \to \mathfrak{g}$ satisfying $J^2 = -1$) is called integrable (Nijenhuis tensor $N(J)$ vanishes) if

$$N(J) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY] = 0, \ \forall X, Y \in \mathfrak{g}.$$

An integrable almost complex structure on the tangent Lie algebra $\mathfrak{g}$ of a real simply connected Lie group G defines a left invariant complex structure on G. If G is nilpotent and $\Gamma \subset G$ is a cocompact lattice, J defines a complex structure on corresponding nilmanifold G/Γ.

We plan to discuss the algebraic constraints on the structure of nilpotent Lie algebra $\mathfrak{g}$ which arise because of the presence of an integrable almost complex structure J on $\mathfrak{g}$.

Salamon studied in [4] 6-dimensional nilpotent Lie algebras admitting integrable complex structure. Goze and Remm have shown [1] that a filiform Lie algebra does not admit any integrable almost complex structure, later Remm and Garcia-Vergnolle extended this result to the class of so-called quasi-filiform Lie algebras [2].

Theorem. *Let $\mathfrak{g}$ be a nilpotent Lie algebra endowed with an integrable complex structure and* $\dim \mathfrak{g} \geq 8$. $\mathfrak{g}^k = [\mathfrak{g}, \mathfrak{g}^{k-1}]$ *denotes k-th ideal of the descending central sequence of the Lie algebra $\mathfrak{g}$. Then we have the following estimates:*

$$\operatorname{codim} \mathfrak{g}^4 \geq 5, \ \operatorname{codim} \mathfrak{g}^6 \geq 8.$$

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We will provide examples showing that these estimates are sharp.

Remark. *For a filiform Lie algebra $\mathfrak{g}$ we have* $\operatorname{codim} \mathfrak{g}^4 = 4$, $\operatorname{codim} \mathfrak{g}^6 = 6$.

[1] Goze M, Remm E. Non existence of complex structures on filiform Lie algebras. Communications in Algebra. 2002; 30(8): 3777-3788.

[2] Garcia-Vergnolle L, Remm E. Complex structures on quasi-filiform Lie algebras. Journal of Lie Theory. 2009; 19(2): 251-265.

[3] Millionshchikov DV. Complex structures on nilpotent Lie algebras and descending central series. http://arxiv.org/abs/1412.0361.
[4] Salamon SM. Complex Structures on Nilpotent Lie Algebras. Journal of Pure and Applied Algebra. 2001; 157: 311-333.