On the existence of pre-semigeodesic coordinates

Irena Hinterleitner

Brno University of Technology, Faculty of Civil Engineering, Brno, CZECH REPUBLIC
[Hinterleitner.I@fce.vutbr.cz]

In the present lecture we consider the problem of the existence of pre-semigeodesic coordinates on manifolds with affine connection. We proved that pre-semigeodesic coordinates exist in the case when the components of the affine connection are differentiable functions.

Let $A_n = (M, \nabla)$ be an $n$-dimensional manifold $M$ with the affine connection $\nabla$, dimension $n \geq 2$, and let $U \subset M$ be a coordinate neighbourhood at the point $x_0 \in U$. A couple $(U, x)$ is a coordinate map on $A_n$.

Semigeodesic coordinate systems on surfaces and (pseudo-) Riemannian manifolds are generalized in the following way [2]:

Definition 1. Coordinates $(U, x)$ in $A_n$ are called pre-semigeodesic coordinates if one system of coordinate lines is geodesic and the natural parameter is just the first coordinate.

In [1,2] the following theorems were proved.

Theorem 1. The conditions $\Gamma^h_{11}(x) = 0$, $h = 1, \ldots, n$, are satisfied in $(U, x)$ if and only if $(U, x)$ is pre-semigeodesic.

We thought that the existence of this chart is trivial. This problem is obviously more difficult than we supposed. This was observed in [3] where precisely the existence of pre-semigeodesic charts was proved in the case when the components of the affine connection are real analytic functions.

We proved that the pre-semigeodesic charts exist in the case when the components of the affine connection are differentiable functions. The following is true:

Theorem 2. For any affine connection determined by $\Gamma^h_{ij}(x) \in C^r(U), r \geq 2$, there exists a local transformation of coordinates determined by $x' = f(x) \in C^r$ such that the connection in the new coordinates $(U', x')$, $U' \subset U$, satisfies $\Gamma'_{11}(x') = 0$, $h = 1, \ldots, n$, i.e. the coordinates $(U', x')$ are pre-semigeodesic and the components $\Gamma'_{ij}(x') \in C^{r-2}(U')$. 
The differentiability class \( r \) is equal to \( 0, 1, 2, \ldots, \infty, \omega \), where \( 0, \infty \) and \( \omega \) denote continuous, infinitely differentiable, and real analytic functions, respectively.

It therefore follows that the existence of a pre-semigeodesic chart is guaranteed in the case when the components of the affine connection \( \nabla \) are twice differentiable. The existence of this chart is not excluded in the case when the components are only continuous.

This is a joint work with Josef Mikeš.