Surfaces with Parallel Normalized Mean Curvature Vector Field in 4-Spaces

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Introduction

Classification results on surfaces with parallel mean curvature vector field:

- Surfaces with parallel mean curvature vector field in Riemannian space forms were classified in [B.-Y. Chen, *Geometry of submanifolds*, 1973] and Yau [S. Yau, *Amer. J. Math.*, 1974].
- Spacelike surfaces with parallel mean curvature vector field in pseudo-Euclidean spaces with arbitrary codimension were classified in [B.-Y. Chen, *J. Math. Phys.* 2009] and [B.-Y. Chen, *Cent. Eur. J. Math.*, 2009].
- Lorentz surfaces with parallel mean curvature vector field in arbitrary pseudo-Euclidean space $\mathbb{E}^m_s$ are studied in [B.-Y. Chen, *Kyushu J. Math.*, 2010] and [Y. Fu, Z.-H. Hou, *J. Math. Anal. Appl.*, 2010].
- A survey on submanifolds with parallel mean curvature vector in Riemannian manifolds as well as in pseudo-Riemannian manifolds is presented in [B.-Y. Chen, *Arab J. Math. Sci.*, 2010].

Introduction

Definition

A submanifold in a Riemannian manifold is said to have ***parallel normalized mean curvature vector field*** if the mean curvature vector is non-zero and the unit vector in the direction of the mean curvature vector is parallel in the normal bundle [B.-Y. Chen, *Monatsh. Math.*, 1980].

Our aim: To describe the surfaces with parallel normalized mean curvature vector field in $\mathbb{E}^4$, in $\mathbb{E}^4_1$, and in $\mathbb{E}^4_2$ in terms of three invariant functions satisfying a system of three partial differential equations.

Our approach: To introduce special geometric parameters on each such surface (***canonical parameters***).

Surfaces in $\mathbb{E}^4$

Let $M^2 : z = z(u, v),\ (u, v) \in \mathcal{D}\ (\mathcal{D} \subset \mathbb{R}^2)$ be a local parametrization of a surface free of minimal points in $\mathbb{E}^4$.

We introduce a geometrically determined moving frame field:

$x = \dfrac{z_u}{\sqrt{E}},\ y = \dfrac{z_v}{\sqrt{G}}$, x and y are collinear with the principal directions;

H – the mean curvature vector field, $b = \dfrac{H}{\sqrt{\langle H, H\rangle}}$; we choose l such that $\{x, y, b, l\}$ is a positively oriented orthonormal frame field in $\mathbb{E}^4$.

Frenet-type derivative formulas:

$$
\begin{array}{ll}
\nabla'_x x = \gamma_1\, y + \nu_1\, b; & \nabla'_x b = -\nu_1\, x - \lambda\, y + \beta_1\, l; \\
\nabla'_x y = -\gamma_1\, x + \lambda\, b + \mu\, l; & \nabla'_y b = -\lambda\, x - \nu_2\, y + \beta_2\, l; \\
\nabla'_y x = -\gamma_2\, y + \lambda\, b + \mu\, l; & \nabla'_x l = -\mu\, y - \beta_1\, b; \\
\nabla'_y y = \gamma_2\, x + \nu_2\, b; & \nabla'_y l = -\mu\, x - \beta_2\, b;
\end{array}
$$

Surfaces in $\mathbb{E}^4$

$\gamma_1,\ \gamma_2,\ \nu_1,\ \nu_2,\ \lambda,\ \mu,\ \beta_1,\ \beta_2$ are functions on M^2 determined by the geometric frame field as follows:

$$\begin{array}{llll} \nu_1 = \langle \nabla'_x x, b \rangle, & \nu_2 = \langle \nabla'_y y, b \rangle, & \lambda = \langle \nabla'_x y, b \rangle, & \mu = \langle \nabla'_x y, l \rangle, \\ \gamma_1 = \langle \nabla'_x x, y \rangle, & \gamma_2 = \langle \nabla'_y y, x \rangle, & \beta_1 = \langle \nabla'_x b, l \rangle, & \beta_2 = \langle \nabla'_y b, l \rangle. \end{array}$$

We call these functions *geometric functions* of the surface since they determine the surface up to a rigid motion in $\mathbb{E}^4$.

For surfaces free of minimal points the function $\mu \neq 0$.

We consider the general class of surfaces for which $\mu_u \, \mu_v \neq 0$.

Fundamental Theorem for surfaces in $\mathbb{E}^4$

Theorem

Let $\gamma_1, \gamma_2, \nu_1, \nu_2, \lambda, \mu, \beta_1, \beta_2$ be smooth functions, defined in a domain $\mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$, and satisfying the conditions

$$\frac{\mu_u}{2\mu\,\gamma_2 + \nu_1\,\beta_2 - \lambda\,\beta_1} > 0; \qquad \frac{\mu_v}{2\mu\,\gamma_1 + \nu_2\,\beta_1 - \lambda\,\beta_2} > 0;$$

$$-\gamma_1\sqrt{E}\sqrt{G} = (\sqrt{E})_v; \qquad -\gamma_2\sqrt{E}\sqrt{G} = (\sqrt{G})_u;$$

$$\nu_1\,\nu_2 - (\lambda^2 + \mu^2) = \frac{1}{\sqrt{E}}(\gamma_2)_u + \frac{1}{\sqrt{G}}(\gamma_1)_v - \left((\gamma_1)^2 + (\gamma_2)^2\right);$$

$$2\lambda\,\gamma_2 + \mu\,\beta_1 - (\nu_1 - \nu_2)\,\gamma_1 = \frac{1}{\sqrt{E}}\lambda_u - \frac{1}{\sqrt{G}}(\nu_1)_v;$$

$$2\lambda\,\gamma_1 + \mu\,\beta_2 + (\nu_1 - \nu_2)\,\gamma_2 = -\frac{1}{\sqrt{E}}(\nu_2)_u + \frac{1}{\sqrt{G}}\lambda_v;$$

$$\gamma_1\,\beta_1 - \gamma_2\,\beta_2 + (\nu_1 - \nu_2)\,\mu = -\frac{1}{\sqrt{E}}(\beta_2)_u + \frac{1}{\sqrt{G}}(\beta_1)_v,$$

Fundamental Theorem for surfaces in $\mathbb{E}^4$

where $\sqrt{E} = \dfrac{\mu_u}{2\mu\,\gamma_2 + \nu_1\,\beta_2 - \lambda\,\beta_1}, \sqrt{G} = \dfrac{\mu_v}{2\mu\,\gamma_1 + \nu_2\,\beta_1 - \lambda\,\beta_2}$. *Let* $\{x_0,\, y_0,\, b_0,\, l_0\}$ *be an orthonormal frame at a point* $p_0 \in \mathbb{R}^4$*. Then there exist a subdomain* $\mathcal{D}_0 \subset \mathcal{D}$ *and a unique surface* $M^2 : z = z(u,v),\ (u,v) \in \mathcal{D}_0$*, passing through* p_0*, such that* $\gamma_1,\, \gamma_2,\, \nu_1,\, \nu_2,\, \lambda,\, \mu,\, \beta_1, \beta_2$ *are the geometric functions of* M^2 *and* $x_0,\, y_0,\, b_0,\, l_0$ *is the geometric frame of* M^2 *at the point* p_0*.*

The meaning of this theorem is that:

Any surface of the general class is determined up to a rigid motion in $\mathbb{E}^4$ by the geometric functions $\gamma_1,\, \gamma_2,\, \nu_1,\, \nu_2,\, \lambda,\, \mu,\, \beta_1, \beta_2$.

Basic classes of surfaces characterized in terms of their geometric functions

The Gauss curvature K, the curvature of the normal connection $K^{\perp}$ and the mean curvature vector field H are expressed as follows:

$$K = \nu_1 \nu_2 - (\lambda^2 + \mu^2); \qquad K^{\perp} = (\nu_1 - \nu_2)\mu; \qquad H = \frac{\nu_1 + \nu_2}{2}\, b.$$

Proposition 1

The surface M^2 is **flat** if and only if $\lambda^2 + \mu^2 = \nu_1 \nu_2$.

Proposition 2

The surface M^2 has **flat normal connection** if and only if $\nu_1 - \nu_2 = 0$.

Proposition 3

The surface M^2 has **non-zero constant mean curvature** if and only if $\nu_1 + \nu_2 = const \neq 0$.

Basic classes of surfaces characterized in terms of their geometric functions

M – n-dimensional submanifold of $(n+m)$-dimensional Riemannian manifold $\widetilde{M}$; ξ – a normal vector field of M.
The *allied vector field* $a(\xi)$ of ξ is defined by the formula

$$a(\xi) = \frac{\|\xi\|}{n} \sum_{k=2}^{m} \{\mathrm{tr}(A_1 \circ A_k)\} \xi_k,$$

where $\{\xi_1 = \dfrac{\xi}{\|\xi\|}, \xi_2, \dots, \xi_m\}$ is an orthonormal base of the normal space of M, and $A_i = A_{\xi_i}, \ i = 1, \dots, m$ is the shape operator with respect to ξ_i.
The allied vector field $a(H)$ of the mean curvature vector field H is called the ***allied mean curvature vector field*** of M in $\widetilde{M}$.

B.-Y. Chen defined the $\mathcal{A}$-submanifolds to be those submanifolds of $\widetilde{M}$ for which $a(H)$ vanishes identically. The $\mathcal{A}$-submanifolds are also called ***Chen submanifolds***.

Basic classes of surfaces characterized in terms of their geometric functions

Proposition 4

The surface M^2 is a **non-trivial Chen surface** if and only if $\lambda = 0$.

Proposition 5

The surface M^2 has **parallel mean curvature vector field** if and only if $\beta_1 = \beta_2 = 0$ and $\nu_1 + \nu_2 = const$.

Proposition 6

The surface M^2 has **parallel normalized mean curvature vector field** if and only if $\beta_1 = \beta_2 = 0$.

Surfaces with parallel normalized mean curvature vector field

Definition

Let M^2 be a surface with parallel normalized mean curvature vector field. The parameters (u, v) of M^2 are said to be ***canonical***, if

$$E(u, v) = \frac{1}{|\mu(u, v)|}; \qquad F = 0; \qquad G(u, v) = \frac{1}{|\mu(u, v)|}.$$

Theorem

Each surface with parallel normalized mean curvature vector field in $\mathbb{E}^4$ locally admits canonical parameters.

Surfaces with parallel normalized mean curvature vector field

Fundamental Theorem – canonical parameters

Let $\lambda(u,v)$, $\mu(u,v)$ and $\nu(u,v)$ be smooth functions, defined in a domain $\mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$, and satisfying the conditions

$$\begin{aligned}
&\mu \neq 0, \quad \nu \neq const;\\
&\nu_u = \lambda_v - \lambda(\ln|\mu|)_v;\\
&\nu_v = \lambda_u - \lambda(\ln|\mu|)_u;\\
&\nu^2 - (\lambda^2 + \mu^2) = \tfrac{1}{2}|\mu|\Delta \ln|\mu|.
\end{aligned}$$

If $\{x_0, y_0, (n_1)_0, (n_2)_0\}$ is a positive oriented orthonormal frame at a point $p_0 \in \mathbb{E}^4$, then there exists a subdomain $\mathcal{D}_0 \subset \mathcal{D}$ and a unique surface $M^2 : z = z(u,v)$, $(u,v) \in \mathcal{D}_0$ with PNMCVF, such that M^2 passes through p_0, the functions $\lambda(u,v)$, $\mu(u,v)$, $\nu(u,v)$ are the geometric functions of M^2 and $\{x_0, y_0, (n_1)_0, (n_2)_0\}$ is the geometric frame of M^2 at the point p_0. Furthermore, (u,v) are canonical parameters of M^2.

Surfaces with parallel normalized mean curvature vector field

Any surface with parallel normalized mean curvature vector field is determined up to a motion in $\mathbb{E}^4$ by three functions $\lambda(u,v)$, $\mu(u,v)$ and $\nu(u,v)$ satisfying the following system of partial differential equations

$$\begin{aligned} \nu_u &= \lambda_v - \lambda(\ln|\mu|)_v; \\ \nu_v &= \lambda_u - \lambda(\ln|\mu|)_u; \\ \nu^2 - (\lambda^2 + \mu^2) &= \tfrac{1}{2}|\mu|\Delta \ln|\mu|. \end{aligned} \tag{1}$$

So, by introducing canonical parameters on a surface with PNMCVF we reduce the number of functions and the number of partial differential equations which determine the surface up to a motion.

Remark: If we assume that $\lambda = 0$, then system (1) implies $\nu = const$. In this case the surface M^2 is of constant mean curvature and has parallel mean curvature vector field. This class of surfaces is described by B.-Y. Chen. So, we assume that $\lambda \neq 0$.

Spacelike surfaces with parallel normalized mean curvature vector field in $\mathbb{E}^4_1$

Any spacelike surface with parallel normalized mean curvature vector field is determined up to a motion in $\mathbb{E}^4_1$ by three functions $\lambda(u,v)$, $\mu(u,v)$ and $\nu(u,v)$ satisfying the following system of partial differential equations

$$\begin{aligned} \nu_u &= \lambda_v - \lambda(\ln|\mu|)_v; \\ \nu_v &= \lambda_u - \lambda(\ln|\mu|)_u; \\ \varepsilon(\nu^2 - \lambda^2 + \mu^2) &= \tfrac{1}{2}|\mu|\Delta \ln|\mu|, \end{aligned} \tag{2}$$

where $\varepsilon = 1$ corresponds to the case the mean curvature vector field H is spacelike, $\varepsilon = -1$ corresponds to the case H is timelike.

Any Lorentz surface of general type with parallel normalized mean curvature vector field is determined up to a rigid motion in $\mathbb{E}^4_2$ by three functions $\lambda(u,v)$, $\mu(u,v)$ and $\nu(u,v)$ satisfying the following system of three partial differential equations

$$\begin{aligned} \nu_u &= -\lambda_v + \lambda(\ln|\mu|)_v; \\ \nu_v &= \lambda_u - \lambda(\ln|\mu|)_u; \\ \varepsilon(\nu^2+\lambda^2-\mu^2) &= \frac{1}{2}|\mu|\Delta^h \ln|\mu|, \end{aligned} \tag{3}$$

where $\Delta^h = \dfrac{\partial^2}{\partial u^2} - \dfrac{\partial^2}{\partial v^2}$ is the hyperbolic Laplace operator;
$\varepsilon = \pm 1 = \mathrm{sign}\langle H, H\rangle$.

Example

Construction of meridian surfaces in $\mathbb{E}^4$:
Let $\{e_1, e_2, e_3, e_4\}$ be the standard orthonormal frame in $\mathbb{E}^4$;
$f = f(u),\ g = g(u)$ – smooth functions, defined in an interval $I \subset \mathbb{R}$, such that $\dot{f}^2(u) + \dot{g}^2(u) = 1,\ u \in I$.

Rotation of the meridian curve $m : u \to (f(u), g(u))$ about the Oe_4-axis:

M^3: $Z(u, w^1, w^2) = f(u)\left(\cos w^1 \cos w^2 e_1 + \cos w^1 \sin w^2 e_2 + \sin w^1 e_3\right) + g(u) e_4.$

Let $w^1 = w^1(v),\ w^2 = w^2(v),\ v \in J,\ J \subset \mathbb{R}$. We consider

$$\mathcal{M} : z(u, v) = Z(u, w^1(v), w^2(v)), \quad u \in I,\ v \in J.$$

$\mathcal{M}$ is a one-parameter system of meridians of the rotational hypersurface M^3. We call $\mathcal{M}$ a ***meridian surface on*** M^3 .

If we denote $l(w^1, w^2) = \cos w^1 \cos w^2\, e_1 + \cos w^1 \sin w^2\, e_2 + \sin w^1\, e_3$, then the parametrization of the rotational hypersurface is written as

$$M^3 : Z(u, w^1, w^2) = f(u)\, l(w^1, w^2) + g(u)\, e_4.$$

Example

The ***meridian surface*** $\mathcal{M}$ is parametrized by:

$$\mathcal{M} : z(u, v) = f(u)\, l(v) + g(u)\, e_4, \quad u \in I,\ v \in J.$$

$l(w^1, w^2)$ is the position vector of the unit 2-dimensional sphere $S^2(1)$ lying in the Euclidean space $\mathbb{E}^3 = \mathrm{span}\{e_1, e_2, e_3\}$ and centered at the origin O; $c : l = l(v) = l(w^1(v), w^2(v)),\ v \in J$ is a smooth curve on $S^2(1)$.

So, each meridian surface is determined by a meridian curve m of a rotational hypersurface and a smooth curve c lying on the sphere $S^2(1)$.

All invariants of $\mathcal{M}$ are expressed by the curvature $\varkappa_m(u)$ of the meridian curve m and the spherical curvature $\varkappa(v)$ of the curve c on $S^2(1)$.

Special cases:

1. If $\varkappa(v) = 0$, i.e. the curve c is a great circle on $S^2(1)$, then the meridian surface $\mathcal{M}$ is a surface lying in a 3-dimensional space.

2. If $\varkappa_m(u) = 0$, i.e. the meridian curve m is part of a straight line, then $\mathcal{M}$ is a developable ruled surface.

Example

We consider meridian surfaces of the general case: $\varkappa_m(u)\,\varkappa(v) \neq 0$.

Geometric functions of $\mathcal{M}$:

$$\gamma_1 = -\gamma_2 = \frac{\dot{f}}{\sqrt{2}f};$$

$$\nu_1 = \nu_2 = \frac{\sqrt{\kappa^2 + (\dot{g} + f\,\kappa_m)^2}}{2f};$$

$$\lambda = \frac{\kappa^2 + \dot{g}^2 - f^2\kappa_m^2}{2f\sqrt{\kappa^2 + (\dot{g} + f\,\kappa_m)^2}};$$

$$\mu = \frac{-\kappa\,\kappa_m}{\sqrt{\kappa^2 + (\dot{g} + f\,\kappa_m)^2}};$$

$$\beta_1 = \frac{1}{\sqrt{2}(\kappa^2 + (\dot{g} + f\,\kappa_m)^2)}\left(\kappa\,\frac{d}{du}\,(\dot{g} + f\,\kappa_m) - \frac{d}{dv}(\kappa)\,\frac{\dot{g} + f\,\kappa_m}{f}\right);$$

$$\beta_2 = -\frac{1}{\sqrt{2}(\kappa^2 + (\dot{g} + f\,\kappa_m)^2)}\left(\kappa\,\frac{d}{du}\,(\dot{g} + f\,\kappa_m) + \frac{d}{dv}(\kappa)\,\frac{\dot{g} + f\,\kappa_m}{f}\right).$$

A solution to the system of PDEs describing the surfaces with PNMCVF

Let $f(u) = \sqrt{u^2 + 2u + 5}$; $\quad g(u) = 2\ln(u + 1 + \sqrt{u^2 + 2u + 5})$;
$c : l = l(v)$ – an arbitrary curve on $S^2(1)$ with $\varkappa(v) \neq 0$.

Geometric functions of $\mathcal{M}$:

$$\gamma_1 = -\gamma_2 = \frac{u+1}{\sqrt{2}(u^2 + 2u + 5)};$$

$$\nu_1 = \nu_2 = \frac{\varkappa(v)}{2\sqrt{u^2 + 2u + 5}};$$

$$\lambda = \frac{\varkappa(v)}{2\sqrt{u^2 + 2u + 5}};$$

$$\mu = \frac{2}{u^2 + 2u + 5};$$

$$\beta_1 = \beta_2 = 0.$$

A solution to the system of PDEs

The mean curvature vector field is

$$H = \frac{\varkappa(v)}{2\sqrt{u^2 + 2u + 5}}\, H_0.$$

Since $\langle H, H\rangle \neq 0$ and $\beta_1 = \beta_2 = 0$, $\mathcal{M}$ is a surface with **parallel normalized mean curvature vector field and non-parallel** H.

Geometric functions λ, μ, ν with respect to the parameters (u, v):

$$\begin{aligned} \lambda &= \frac{\varkappa(v)}{2\sqrt{u^2 + 2u + 5}}; \\ \mu &= \frac{2}{u^2 + 2u + 5}; \\ \nu &= \frac{\varkappa(v)}{2\sqrt{u^2 + 2u + 5}}; \end{aligned} \tag{4}$$

The parameters (u, v) coming from the parametrization of the meridian curve m are NOT canonical parameters of $\mathcal{M}$.

A solution to the system of PDEs

Change of the parameters:

$$\begin{aligned} \bar{u} &= \ln(u + 1 + \sqrt{u^2 + 2u + 5}) + v \\ \bar{v} &= \ln(u + 1 + \sqrt{u^2 + 2u + 5}) - v \end{aligned} \tag{5}$$

$(\bar{u}, \bar{v})$ – canonical parameters of $\mathcal{M}$.

Hence, the functions $\lambda(u(\bar{u}, \bar{v}), v(\bar{u}, \bar{v}))$, $\mu(u(\bar{u}, \bar{v}), v(\bar{u}, \bar{v}))$, and $\nu(u(\bar{u}, \bar{v}), v(\bar{u}, \bar{v}))$ give a solution to the following system of PDEs

$$\begin{aligned} \nu_{\bar{u}} &= \lambda_{\bar{v}} - \lambda(\ln|\mu|)_{\bar{v}}; \\ \nu_{\bar{v}} &= \lambda_{\bar{u}} - \lambda(\ln|\mu|)_{\bar{u}}; \\ \nu^2 - (\lambda^2 + \mu^2) &= \tfrac{1}{2}|\mu| \Delta \ln|\mu|. \end{aligned} \tag{6}$$

A solution to the system of PDEs

Solutions to the system

$$\begin{aligned} \nu_u &= \lambda_v - \lambda(\ln|\mu|)_v; \\ \nu_v &= \lambda_u - \lambda(\ln|\mu|)_u; \\ \varepsilon(\nu^2 - \lambda^2 + \mu^2) &= \tfrac{1}{2}|\mu|\Delta \ln|\mu|, \end{aligned}$$

and the system

$$\begin{aligned} \nu_u &= -\lambda_v + \lambda(\ln|\mu|)_v; \\ \nu_v &= \lambda_u - \lambda(\ln|\mu|)_u; \\ \varepsilon(\nu^2 + \lambda^2 - \mu^2) &= \frac{1}{2}|\mu|\Delta^h \ln|\mu|, \end{aligned}$$

can be found in the class of meridian surfaces in $\mathbb{E}^4_1$ and $\mathbb{E}^4_2$, respectively. Note that in $\mathbb{E}^4_1$ there exist three types of spacelike meridian surfaces, in $\mathbb{E}^4_2$ there are four types of Lorentz meridian surfaces. All of them give solutions to the corresponding system of PDEs.

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Thank you for your attention!