THE SAMPSON LAPLASIAN ACTING ON COVARIANT SYMMETRIC TENSORS

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1. Introduction

Forty five years ago J. H. Sampson has defined a Laplacian operator Δ_S acting on covariant symmetric tensors [1]. This operator was an analogue of the well known Hodge-de Rham Laplacian Δ_H which acts on exterior differential forms [2]. These two operators Δ_S and Δ_H are self-adjoint elliptic operators and hence their kernels are finite-dimensional vector spaces on a compact Riemannian manifold. In addition, the Sampson operator Δ_S admits the Weitzenböck decomposition formula as well as the Hodge-de Rham Laplacian Δ_H.

[1] Sampson J. H., On a theorem of Chern, Trans. Amer. Math. Soc., 177 (1973), 141-153.

[2] Petersen P., Riemannian geometry, Springer Science, New York (2006).

In our report, we will consider the little-known Sampson Laplacian Δ_S using the analytical method, due Bochner, of proving vanishing theorems for the null space of a Laplace operator admitting a Weitzenböck decomposition (see [1]; [2]) and further of estimating its lowest eigenvalue (see, for example, [3]).

[1] Bérard P.H., From vanishing theorems to estimating theorems: the Bochner technique revisited, Bulletin of American Mathematical Society, 19:2 (1988), 371-406.

[2] Pigola, S., Rigoli, M., Setti, A.G.: Vanishing and finiteness results in geometric analysis. A generalization of the Bochner technique, Birkhäuser, Basel (2008).

[3] Craioveanu M., Puta M., Rassias T. M., Old and new aspects in spectral geometry, Kluwer Academic Publishers, London (2001).

Theorems and corollaries of the report complement our results of our paper from the following list: [1]; [2]; [3]; [4]. In addition, applications of Sampson Laplacian Δ_S can be find in our paper [1] and [5].

[1] Stepanov S.E., Mikeš J., The spectral theory of the Yano rough Laplacian with some of its applications, Ann. Glob. Anal. Geom., 48 (2015), 37-46.

[2] Stepanov S.E., Vanishing theorems in affine, Riemannian and Lorentz geometries, Journal of Mathematical Sciences (New York), 141:1 (2007), 929-964.

[3] Stepanov S.E., Tsyganok I.I., Mikesh J., On a Laplacian which acts on symmetric tensors, Preprint, arXiv: 1406.2829 [math.DG], 1 (2014), 14 pp.

[4] Stepanov S.E., Tsyganok I.I., Aleksandrova I.A., A remark on the Laplacian operator which acts on symmetric tensors, Preprint, arXiv: 1411.1928 [math.DG], 4 (2014), 8 pp.

[5] Mikeš J., Stepanova E.S., A five-dimensional Riemannian manifold with an irreducible SO(3)-structure as a model of abstract statistical manifold, Ann. Glob. Anal. Geom., 45:2 (2014), 111-128.

2. Preliminaries

Let (M, g) be a Riemannian manifold of dimension $n \geq 2$ with the Levi-Civita connection ∇. Let TM (resp. T^*M) be its tangent (resp. cotangent) bundle, and let $S^pM = S^p\left(T^*M\right)$ be the bundles of covariant symmetric p-tensors on M. The formula

$$\langle \varphi , \psi \rangle = \frac{1}{p!} \int_M g(\varphi , \psi) \, dv_g , \qquad (2.1)$$

where $\varphi, \psi \in C^\infty S^pM$ and dv_g is the volume element of (M, g) determines the $L^2(M, g)$-scalar product on $C^\infty S^pM$.

We will apply the above to the operator

$$\delta^* : C^\infty S^p M \to C^\infty S^{p+1} M$$

of degree 1 such that $\delta^* = (p+1)\ \mathrm{Sym}\nabla$ where $\mathrm{Sym} : \otimes^p T^* M \to S^p M$ is the linear operator of symmetrization. Then there exists its formal adjoint operator

$$\delta : C^\infty S^{p+1} M \to C^\infty S^p M$$

with respect to the $L^2(M, g)$-product that is called the divergence operator (see [1, p. 55; 356]).

[1] Besse A.L, Einstein manifolds, Springer-Verlag, Berlin – Heidelberg 1987.

Sampson has defined in [1, p. 147] the second order operator

$$\Delta_S = \delta\delta^* - \delta^*\delta : \ C^\infty S^p M \to C^\infty S^p M$$

for an arbitrary Riemannian manifold (M, g). Moreover, it was shown in [1, p. 147] that the operator Δ_S has the Weitzenböck decomposition

$$\Delta_S = \overline{\Delta} - \Gamma_p \tag{2.2}$$

where Γ_p can be algebraically (even linearly) expressed through the curvature R and Ricci Ric tensors of (M, g) and $\overline{\Delta} = \nabla^*\nabla$ is the Bochner Laplacian (see [2, pp. 53; 356]).

[1] Sampson J. H., On a theorem of Chern, Transactions of the American Mathematical Society, 177 (1973), 141-153.

[2] Besse A.L, Einstein manifolds, Springer-Verlag, Berlin – Heidelberg (1987).

Remark. The Sampson operator can be found in the monograph [1, p. 356] and in the papers from the following list [2]; [3]; [4]. But in fact, we were the first and only who began to study the properties of this operator in details.

[1] Besse A.L, Einstein manifolds, Springer-Verlag, Berlin – Heidelberg 1987.

[2] Sumitomo T., Tandai K., Killing tensor fields on the standard sphere and spectra of $SO(n+1)/(SO(n-1)\times SO(2))$ and $O(n+1)/(O(n-1)\times O(2))$, Osaka Journal of Mathematics, 20 : 1 (1983), 51-78.

[3] Boucetta M., Spectre des Laplaciens de Lichnerowicz sur les sphères et les projectifs réels, Publicacions Matemàtiques, 43 (1999), 451-483.

[4] Heil K., Moroianu A., Semmelmann U., Killing and conformal Killing tensors, J. Geom. Phys., 106 (2016), 383-400.

The following properties are the elementary properties of Sampson operator Δ_S on a compact Riemannian manifold (M,g).

(i) The operator Δ_S is a self-adjoint operator with respect to the $L^2(M,g)$-product, i.e. $\langle \Delta_S \varphi, \psi \rangle = \langle \varphi, \Delta_S \psi \rangle$ for any $\varphi, \psi \in C^\infty S^p M$.

(ii) The principal symbol σ of Δ_S satisfies the condition $\sigma(\Delta_S)(\theta, x)\varphi_x = -g(\theta,\theta)\varphi_x$ for an arbitrary $\theta \in T_x^* M - \{0\}$. Therefore, by the Sampson operator Δ_S is a Laplacian and its kernel is a finite-dimensional vector space on a compact manifold (M,g).

(iii) Two vector spaces $\operatorname{Ker}\Delta_S$ and $\operatorname{Im}\Delta_S$ are orthogonal complements of each other with respect to the $L^2(M,g)$-product, i.e.

$$C^\infty S^p M = \operatorname{Ker}\Delta_S \oplus \operatorname{Im}\Delta_S .$$

2. The kernel of the Sampson Laplacian

Let (M, g) be a locally Euclidean manifold then the equation $\Delta_S \varphi = 0$ is equivalent to the equation $\sum_k \frac{\partial^2 \varphi_{i_1 \ldots i_p}}{(\partial x^k)^2} = 0$ with respect to a local Cartesian coordinate system $x^1, \ldots, x^n$. This means that all components of this tensor φ are harmonic functions. Therefore, the symmetric tensor $\varphi \in ker\, \Delta_S$ was named in [1, p. 148] as a harmonic symmetric p-tensor on (M, g).

[1] Sampson J. H., On a theorem of Chern, Transactions of the American Mathematical Society, 177 (1973), 141-153.

The “energy” of symmetric tensor field φ is given by the formula $E(\varphi)=$ ½$\langle\varphi,\Delta_S\varphi\rangle$, then the equation $\Delta_S\varphi=0$ is the condition for a free extremal of $E(\varphi)$ for an arbitrary compact (M, g) (see [1, p. 148]).

In addition, in [1, p. 151] was proved that for a compact Riemannian manifold of constant negative curvature the only harmonic non-zero p-tensor fields are those of the form $const\times\mathrm{Sym}(g\otimes g\otimes\ldots\otimes g)$.

Other a non-trivial interesting example of a harmonic symmetric tensor can be found in our paper [2].

[1] Sampson J. H., On a theorem of Chern, Transactions of the American Mathematical Society, 177 (1973), 141-153.

[2] Mikeš J., Stepanova E.S., A five-dimensional Riemannian manifold with an irreducible SO(3)-structure as a model of abstract statistical manifold, Ann. Glob. Anal. Geom., 45:2 (2014), 111-128.

We recall that the tensor field $\varphi \in C^{\infty} S^{p} M$ which satisfies the equation $\delta^{*}\varphi = 0$ is well known in the theory of general relativity as a symmetric Killing tensor (see, for example, [1] and [2]). Then an arbitrary a divergence-free symmetric Killing p-tensor φ belongs to $\operatorname{Ker} \Delta_{S}$.

It is easy to verify that an arbitrary trace-free symmetric Killing p-tensor φ is a divergence-free symmetric Killing p-tensor. Therefore, an arbitrary trace-free symmetric Killing p-tensor φ belongs to $\operatorname{Ker} \Delta_{S}$.

[1] Collinson C.D., Howarth L., Generalized Killing tensors, General Relativity and Gravitation, 32:9 (2000), 1767-1776.

[2] Dolan P., Kladouchou A., Card C., On the significance of Killing tensors, General Relativity and Gravitation, 21:4 (1989), 427-437.

Theorem 2.1. Let φ be a divergence-free (or trace-free) symmetric Killing tensor on a Riemannian manifold (M, g), then it satisfies the following systems of differential equations

(i) $$\Delta_S \varphi = 0;$$

(ii) $$\delta \varphi = 0.$$

Conversely, if (M, g) is compact and a tensor field $\varphi \in C^\infty S^p M$ satisfies (i) and (ii), then φ is a divergence-free Killing tensor.

Remark. For $p = 1$, from Theorem 2.1 we obtain Theorem 2.3 on infinitesimal isometrics presented in Kobayashi's monograph on transformation groups (see [1]).

[1] Kobayashi S., Transformation groups in differential geometry, Springer-Verlag, Berlin and Heidelberg (1995).

For the case $p = 1$, the Sampson Laplacian can be rewriten in the form $\Delta_S = \overline{\Delta} - \Gamma_1$ where $\Gamma_1 = Ric$ for the Ricci tensor Ric of (M, g). Therefore, we have the following theorem (see [1]).

Theorem 2.2. the Sampson Laplacian $\Delta_S : C^\infty T^* M \to C^\infty T^* M$ is dual to the Yano Laplacian □: $C^\infty TM \to C^\infty TM$ by the metric g.

Remark. The operator □: $C^\infty TM \to C^\infty TM$ was defined by Yano for the investigation of local isometric, conformal, affine and projective transformations of compact Riemannian manifolds (see [2, p. 40]).

[1] Stepanov S.E., Mikeš J., The spectral theory of the Yano rough Laplacian with some of its applications, Ann. Glob. Anal. Geom., 48 (2015), 37-46.

[2] Yano K., Integral formulas in Riemannian geometry, Marcel Dekker, New York (1970).

The vector field ξ on (M,g) is called an infinitesimal harmonic transformation if the one-parameter group $\psi\,.\,(t,x)\in\mathbb{R}\times M\to\psi_t(x)\in M$ of infinitesimal point transformations of (M,g) generated by ξ consists of harmonic diffeomorphisms (see [1]). We have proved in [2] that the following theorem is true.

Theorem 2.3. Vector field ξ is an infinitesimal harmonic transformation on (M,g) if and only if $\Delta_S\,\varphi=0$ for the 1-form φ corresponding to ξ under the duality defined by the metric g.

[1] Stepanov S.E., Shandra I.G., Geometry of infinitesimal harmonic transformations, Ann. Glob. Anal. Geom., 24 (2003), 291-299.

[2] Stepanov S.E., Mikeš J., The spectral theory of the Yano rough Laplacian with some of its applications, Ann. Glob. Anal. Geom., 48 (2015), 37-46.

We have proved also that a Killing vector on a Riemannian manifold, holomorphic vector field on a nearly Kählerian manifold and the vector field that transforms a Riemannian metric into a Ricci soliton metric are examples of infinitesimal harmonic transformations (see [1]; [2]). Therefore, all one-forms which corresponding to these vector fields under the duality defined by the metric g belong to the kernel for the Sampson Laplacian Δ_S.

[1] Stepanov S.E., Shandra I.G., Geometry of infinitesimal harmonic transformations, Ann. Glob. Anal. Geom., 24 (2003), 291-299.

[2] Stepanov S.E., Mikeš J., The spectral theory of the Yano rough Laplacian with some of its applications, Ann. Glob. Anal. Geom., 48 (2015), 37-46.

Let ω be an arbitrary one-form such that $\omega \in \operatorname{Ker} \Delta_S$. In accordance with the theory of harmonic maps (see [1]) we define the energy density of the flow on (M, g) generated by the vector field $\xi = \omega^{\#}$ as the scalar function $e(\xi) = \frac{1}{2} \| \xi \|^2$ where $\| \xi \|^2 = g(\xi, \xi)$. Then the Beltrami Laplacian $\Delta_B\, e(\xi) := \operatorname{div}(\operatorname{grad} e(\xi))$ for the energy density $e(\xi)$ of an infinitesimal harmonic transformation $\xi = \omega^{\#}$ has the form (see [2])

$$\Delta_B\, e(\xi) = \| \nabla \omega \|^2 - Ric(\xi, \xi). \tag{2.3}$$

[1] Eells, J., Sampson, J.H., Harmonic mappings of Riemannian manifolds, American Journal of Mathematics, 86 (1964), no. 1, 109-160.

[2] Stepanov S.E., Mikeš J., The spectral theory of the Yano rough Laplacian with some of its applications, Ann. Glob. Anal. Geom., 48 (2015), 37-46.

We recall that the Ricci curvature of g is quasi-negative if it is non-negative everywhere in a connected open domain $U \subset M$ and it is strictly negative in all directions at some point of U. In this case, $e(\xi)$ is a subharmonic function. Then using the Hopf's maximum principle (see [1]), we can prove the following

Theorem 2.3. Let (M, g) be a Riemannian manifold and $U \subset M$ be a connected open domain with the quasi-negative Ricci tensor Ric. If the energy density of the flow $e(\xi) = \frac{1}{2} \| \xi \|^2$ generated by $\xi = \omega^{\#}$ for an arbitrary one-form $\omega \in \mathrm{Ker}\, \Delta_S$ has a local maximum in some point of U, then ω is identically zero everywhere in U.

[1] Calabi E., An extension of E. Hopf's maximum principle with an application to Riemannian geometry, Duke Math. J., 25 (1957), 45-56.

Remark. Theorem 2.3. is a direct generalization of the Theorem 4.3 presented in Kobayashi's monograph on transformation groups (see [1, p. 57]) and Wu's proposition on a Killing vector whose length achieves a local maximum (see [2]).

In addition, we can formulate the following statement, which is a corollary of Theorem 2.3.

Corollary 2.4. The Sampson Laplacian $\Delta_S : C^\infty T^*M \to C^\infty T^*M$ has a trivial kernel on a compact Riemannian manifold (M, g) with quasi-negative Ricci curvature.

[1] Kobayashi S., Transformation groups in differential geometry, Springer-Verlag, Berlin and Heidelberg, 1995.
[2] Wu H., A remark on the Bochner technique in differential geometry, Proc. Amer. Math. Soc., 78:3 (1980), 403-408.

Let $\Delta_S : C^\infty S^2M \to C^\infty S^2M$ be the Sampson Laplacian acting on the vector space of covariant symmetric 2-tensors S^2M. In this case, the Weitzenböck decomposition formula (2.2) has the form $\Delta_S\,\varphi = \overline{\Delta}\varphi - \Gamma_2(\varphi)$ where $\Gamma_2(\varphi) = \left(R_{ik}\varphi_j^k + R_{jk}\varphi_i^k\right) - 2R_{ikjl}\varphi^{kl}$ for the local components φ_{ij} of $\varphi \in C^\infty S^2M$, R_{ij} of the Ricci tensor and R_{ijkl} of the curvature tensor R. Let $\varphi \in \operatorname{Ker}\Delta_S$, then direct calculations give us the formula

$$\frac{1}{2}\Delta_B \left\| \varphi \right\|^2 = \left\| \nabla \varphi \right\|^2 - 2\cdot\sum_{i<j} \sec\left(e_i \wedge e_j\right)\left(\mu_i - \mu_j\right)^2 \qquad (2.4)$$

for a local orthonormal frame $e_1,\dots,e_n$ such that $\varphi_x\left(e_i, e_j\right) = \mu_i\,\delta_{ij}$ and for the sectional curvature $\sec\left(e_i \wedge e_j\right)$ in the two-direction $e_i \wedge e_j$.

Then using the Hopf's maximum principle (see [1]), we can prove the following

Theorem 2.4. Let U be a connected open domain of a Riemannian manifold (M, g), φ be a 2-tensor field defined on U such that $\varphi \in \mathrm{Ker}\,\Delta_S$ everywhere in U. If the section curvature of (M, g) is negative semi-define at any point of U and the scalar function $\|\varphi\|^2$ has a local maximum at some point of U, then φ is invariant under parallel translation in U, i.e. $\nabla\varphi = 0$. If, moreover, $\sec < 0$ at some point of U or (M, g) is an irreducible Riemannian manifold, then φ is constant multiple of g at all points of U.

[1] Calabi E., An extension of Hopf's maximum principle with an application to Riemannian geometry, Duke Math. J., 25 (1957), 45-56.

Based on (2.4) and the Bochner maximum principle (see [1, Theorem 2.2]), we can formulate the statement that is a corollary of our Theorem 2.4.

Corollary 2.5. Let (M, g) be a compact Riemannian manifold (M, g) with nonpositive sectional curvature, then the kernel of the Sampson Laplacian $\Delta_S : C^\infty S^2 M \to C^\infty S^2 M$ consists of parallel symmetric 2- tensor fields. If the sectional curvature in all directions is less than zero at some point of (M, g) or (M, g) is an irreducible Riemannian manifold, then an arbitrary tensor field which belongs to the kernel of the Sampson Laplacian is constant multiple of g.

[1] Bochner S., Yano K., Curvature and Betti numbers, Princeton, Princeton University Press (1953).

We recall here that a complete simply connected nonpositively curved manifold (M, g) is called a Hadamar manifold (see [1, p. 381]). In particular, a Riemannian (globally) symmetric manifold of the non-compact type is a non-trivial example of a Hadamard manifold (M, g), since it is a simply connected Riemannian symmetric manifold with nonpositive (but not identically zero) sectional curvature (see [2, pp. 256; 258]). Based on (2.4) and the Yau theorem on subharmonic function (see [3, p. 663]), we can formulate

[1] Li P., Geometric Analysis, Cambridge University Press, Cambridge, 2012.
[2] Kobayashi Sh., Nomizu K., Foundations of differential geometry, vol. II, New York-London-Sydney, Int. Publishers (1969).
[3] Yau S.-T., Some function-theoretic properties of complete Riemannian manifold and their applications to geometry, Indiana Univ. Math. J., 25:7 (1976), 659-670.

Corollary 3.3. Let φ be a symmetric 2-tensor on a Hadamar manifold, in particular on a Riemannian symmetric manifold (M, g) of the non-compact type. If $\varphi \in \operatorname{Ker} \Delta_S$ and $\int_M \|\varphi\|^q \, dVol_g < +\infty$ at least for one $q \geq 1$. Then φ is invariant under parallel translation, i.e., $\nabla \varphi = 0$. If in this case the volume of (M, g) is infinite, then the harmonic symmetric 2-tensor φ is identically zero.

Remark. In [3, p. 663] was proved the following theorem: Let u be a nonnegative subharmonic function on a complete manifold (M, g), then $\int_M u^q \, dv_g = \infty$ for $q > 1$, unless u is a constant function C.

3. Spectral properties of the Sampson Laplacian

A real number λ^p, for which there is a symmetric p-tensor $\varphi \in C^\infty S^p M$ (not identically zero) such that $\Delta_S \varphi = \lambda^p \varphi$, is called an eigenvalue of the Sampson Laplacian $\Delta_S : C^\infty S^p M \to C^\infty S^p M$ and the corresponding symmetric p-tensor $\varphi \in C^\infty S^p M$ is called an eigentensor of the Sampson Laplacian Δ_S corresponding to λ^p. All nonzero eigentensors corresponding to a fixed eigenvalue λ^p form a vector subspace of $S^p M$ denoted by $V_{\lambda^p}(M)$ and called the eigenspace of the Sampson Laplacian corresponding to its eigenvalue λ^p.

Using the general theory of elliptic operators on a compact Riemannian manifold (M, g) it can be proved that Δ_S has a discrete spectrum, denoted by $\mathrm{Spec}^{(p)}\,\Delta_S$, consisting of real eigenvalues of finite multiplicity which accumulate only at infinity (see [1]). In symbols, we have

$$\mathrm{Spec}^{(p)}\,\Delta_S = \left\{0 \le \left|\lambda_1^p\right| \le \left|\lambda_2^p\right| \le \ldots \to +\infty\right\}.$$

In addition, if we suppose that $\Delta_S : C^\infty T^*M \to C^\infty T^*M$ and the Ricci tensor Ric is negative everywhere on (M, g) then (see [2])

$$\mathrm{Spec}^{(1)}\,\Delta_S = \left\{0 < \lambda_1^1 \le \lambda_2^1 \le \ldots \to +\infty\right\}.$$

[1] Craioveanu M., Puta M., Rassias T. M., Old and new aspects in spectral geometry, Kluwer Academic Publishers, London (2001).

[2] Stepanov S.E., Mikeš J., The spectral theory of the Yano rough Laplacian with some of its applications, Ann. Glob. Anal. Geom., 48 (2015), 37-46.

Theorem 3.1. Let (M, g) be an n-dimensional $(n \geq 2)$ compact and oriented Riemannian manifold and Δ_S: $C^{\infty}T^{*}M \to C^{\infty}T^{*}M$ be the Sampson Laplacian.

(i) Suppose the Ricci tensor is negative then an arbitrary eigenvalue λ^{1} of Δ_S is positive.

(ii) The eigenspaces of Δ_S are finite dimensional.

(iii) The eigentensors corresponding to distinct eigenvalues are orthogonal.

Theorem 3.2. Let (M, g) be a 2-dimensional compact oriented Riemannian manifold. Then the first eigenvalue λ_1^1 of the Sampson Laplacian Δ_S: $C^{\infty}T^{*}M \to C^{\infty}T^{*}M$ is a non-negative number.

Moreover, the following theorem is true (see [1]).

Theorem 3.3. Let (M, g) be an n-dimensional ($n \geq 2$) compact oriented Riemannian manifold. Suppose the Ricci tensor Ric is negative, then the first eigenvalue λ_1^1 of the Sampson Laplacian Δ_S: $C^\infty T^*M \to C^\infty T^*M$ satisfies the inequality $\lambda_1^1 \geq 2r$ for the largest (negative) eigenvalue $-r$ of the Ricci tensor Ric on (M, g). The equality $\lambda_1^1 = 2r$ is attained for some harmonic eigenform $\varphi \in C^\infty T^*M$ and in this case the multiplicity of λ_1^1 is less than or equals to the Betti number $b_1(M)$.

[1] Stepanov S.E., Mikeš J., The spectral theory of the Yano rough Laplacian with some of its applications, Ann. Glob. Anal. Geom., 48 (2015), 37-46.

We consider now the Sampson Laplacian $\Delta_S : C^\infty S^2 M \to C^\infty S^2 M$. It has a discrete spectrum, denoted by

$$\mathrm{Spec}^{(2)}\, \Delta_S = \left\{ 0 \le \left|\lambda_1^2\right| \le \left|\lambda_2^2\right| \le \ldots \to +\infty \right\}.$$

Theorem 3.4. Let (M, g) be an n-dimensional $(n \ge 2)$ compact and oriented Riemannian manifold and $\Delta_S : C^\infty S^2 M \to C^\infty S^2 M$ be the Sampson Laplacian. Suppose the section curvature is negative defined then an arbitrary eigenvalue λ_α^2 of Δ_S is positive and

$$\mathrm{Spec}^{(2)}\, \Delta_S = \left\{ 0 < \lambda_1^2 \le \lambda_2^2 \le \ldots \to +\infty \right\}.$$

.

Let $(M,\ g)$ be an n-dimensional compact and oriented Riemannian manifold with sectional curvature bounded above by a strictly negative constant $-k$. Then the following theorem holds.

Theorem 3.5. Let $(M,\ g)$ be an n-dimensional compact and oriented Riemannian manifold with sectional curvature bounded above by a strictly negative constant $-k$. Then any eigenvalue λ^2 of the Sampson Laplacian $\Delta_S : C^\infty S^2M \to C^\infty S^2M$ satisfies the inequality $\lambda^2 \geq n\,k$ for the non-zero eigentensor $\varphi \in C^\infty S^2M$ such that φ corresponds to the eigenvalue λ^2 and $\varphi \neq \mu\, g$ for some smooth scalar function μ.

Next we will consider the Sampson Laplacian $\Delta_S : C^\infty S_0^2 M \to C^\infty S_0^2 M$ acting on the smooth sections of the vector bundle of trace-free symmetric 2-tensor fields $S_0^2 M$ on a compact Riemannian manifold (M, g). The following obvious statement is true.

Theorem 3.6. The Sampson Laplacian Δ_S maps $S_0^2 M$ to itself.

Let (M, g) be a compact Riemannian manifold with negative sectional curvature. We denote by $K_{\max}$ the maximum of the section curvature of (M, g). In particular, we can consider a compact hyperbolic manifold $(\mathbb{H}^n, g_0)$ with constant sectional curvature equal to -1. In this case, the first eigenvalue λ_1^2 of the Sampson Laplacian defined on trace-free symmetric 2-tensor fields satisfies the inequalitiy $\lambda_1^2 \geq 2n$.

This proposition is a corollary of the following theorem.

Theorem 3.7. Let (M, g) be an n-dimensional $(n \geq 2)$ compact Riemannian manifold with negative sectional curvature and $\Delta_S : C^\infty S_0^2 M \to C^\infty S_0^2 M$ be the Sampson Laplacian acting on trace-free symmetric 2-tensor fields. Then the first eigenvalue of Δ_S satisfies the inequality $\lambda_1^2 \geq -2n\, K_{\max}$ for the maximum $K_{\max}$ of the sectional curvature of (M, g). If $\lambda_1^2 = -2n\, K_{\max}$, then the trace-free symmetric 2-tensor field φ corresponding to λ_1^2 is invariant under parallel translation. In this case, if (M, g) is irreducible then φ is a constant multiplied by the metric g at each point of (M, g).

Thank a lot for your attention!