About almost geodesics curves

Olga Belova\textsuperscript{1}
Josef Mikeš\textsuperscript{2}
Karl Strambach\textsuperscript{3}

\textsuperscript{1} Immanuel Kant Baltic Federal University, Kaliningrad, Russia
\textsuperscript{2} Palacky University, Olomouc, Czech Republic
\textsuperscript{3} Erlangen-Nürnberg University, Erlangen, Germany

Vrnjačka Banja, Serbia, May 20–23, 2018
Abstract

We determine in $\mathbb{R}^n$ the form of curves $C$ for which also any image under an $(n - 1)$-dimensional algebraic torus is an almost geodesic with respect to an affine connections $\nabla$ with constant coefficients and calculate explicitly the components of $\nabla$. 
1. Introduction

Geodesics are classical objects of differential geometry. They are invariants for geodesic mappings.

E. Beltrami has shown that a differentiable curve is a local geodesic with respect to an affine connection $\nabla$ precisely if it is a solution of an Abelian differential equation with coefficients which are functions of the components of $\nabla$.

Almost geodesics curves and mappings have been introduced in 1963 by N.S. Sinyukov as generalizations of geodesic curves and mappings.
2. Introduction

The explicit calculation of the form of curves $C$ in the $n$-dimensional real space $\mathbb{R}^n$ which are geodesics or almost geodesics with respect to an affine connection $\nabla$ is not achievable even in the case if the components $\Gamma^h_{ij}$ of $\nabla$ are constant. But we can do it if we suppose that with $C$ also all images of $C$ under a real $(n - 1)$-dimensional algebraic torus are also almost geodesics. The determination of $C$ becomes an algebraic problem, namely a problem of polynomial identities.
We consider a curve $C$ homeomorphic to $\mathbb{R}$ which is a closed subset of $\mathbb{R}^n$ and has the form

$$C = \left(t, f_2(t), \ldots, f_n(t)\right), \quad t \in \mathbb{R},$$

(1)

where $f_i(t): \mathbb{R} \to \mathbb{R}$, $i = 2, \ldots, n$, are three times differentiable non-constant functions.

The system

$$\mathcal{X}(C) = \left\{ \left(t + c_1, b_2f_2(t) + c_2, \ldots, b_n f_n(t) + c_n \right), \quad t \in \mathbb{R} \right\},$$

(2)

where $b_i \neq 0$, $c_i \in \mathbb{R}$,

is a set of imagines of $C$. 
✓ If every curve of $\mathcal{X}(C)$ is a geodesic with respect to an affine connection $\nabla$ with constant coefficients $\Gamma^h_{ij}$, then the derivatives $f'_i(t)$ of the functions $f_i(t)$ are solutions of the first order linear ordinary differential equations.

✓ If every curve of $\mathcal{X}(C)$ is an almost geodesic with respect to $\nabla$, then the derivatives $f'_i(t)$ are solutions of harmonic oscillator equations.

✓ If $\mathcal{X}(C)$ consists of Euclidean lines which are geodesic with respect to $\nabla$, then at the most $\Gamma^1_{11}$ may be different from 0.

In contrast to this

✓ if $\mathcal{X}(C)$ consists of Euclidean lines then there is huge quantity of non-trivial connections $\nabla$ such that the lines of $\mathcal{X}(C)$ are almost geodesic with respect to $\nabla$. 

---

Immanuel Kant Baltic Federal University, Kaliningrad, Russia
Palacky University, Olomouc, Czech Republic
Erlangen-Nürnberg University, Erlangen, Germany

About almost geodesics curves
We apply results of differential geometry only for the $n$-dimensional space $\mathbb{R}^n$, where global coordinates exist. The components 

$$\Gamma^h_{ij}, \quad h, i, j \in \{1, 2, \ldots, n\},$$

of any affine connection $\nabla$ can be written in unique way in these coordinates.
Let
\[ \ell = \left( t + c_1, b_2 f_2(t) + c_2, \ldots, b_n f_n(t) + c_n \right), \quad t \in \mathbb{R}, \quad (3) \]
be a curve of \( \mathfrak{X}(\mathcal{C}) \).

Then
\[ \dot{\ell} = \left( 1, b_2 f_2'(t), \ldots, b_n f_n'(t) \right), \quad (4) \]
\[ \ddot{\ell} = \left( 0, b_2 f_2''(t), \ldots, b_n f_n''(t) \right). \]
7. Geodesics

**Definition**

By a *geodesic* of $\nabla$ we mean a piecewise $C^2$-curve $\gamma: I \to \mathbb{R}^n$ satisfying

$$\nabla \dot{\gamma} \ddot{\gamma} = \varrho \cdot \dot{\gamma}, \tag{5}$$

where $\varrho: I \to \mathbb{R}$ is a continuous function, $I \subset \mathbb{R}$ is an open interval.

Using the components of $\nabla$ the system of differential equations for geodesics has the form

$$\ddot{\gamma}^h + \sum_{i,j=1}^{n} \Gamma_{ij}^h \dot{\gamma}^i \dot{\gamma}^j = \varrho(t) \dot{\gamma}^h, \quad h \in \{1, 2, \ldots, n\}, \tag{6}$$

where $\Gamma_{ij}^h$ are constant coefficients of an affine connections $\nabla$. 

About almost geodesics curves
8. Geodesics

According to the (6) a curve $\ell$ of $\mathcal{X}(\mathcal{C})$ is a geodesic with respect to $\nabla$ if and only if

$$\ddot{\ell}_h + \sum_{i,j=1}^{n} \Gamma^h_{ij} \dot{\ell}_i \dot{\ell}_j = \varrho(t) \dot{\ell}_h, \quad h = 1, \ldots, n. \quad (7)$$

From this for $h = 1$ we obtain $\varrho(t)$ and for $h = 2, \ldots, n$ after substitution of the function $\varrho(t)$ we get equations (8)
9. Geodesics

\[ \Gamma^h_{11} + b_h \cdot \left( f''_h(t) + 2\Gamma^h_{1h} \cdot f'_h(t) \right) + b_h^2 \cdot \left( \Gamma^h_{hh} - 2\Gamma^1_{1h} \right) (f'_h(t))^2 - b_h^3 \cdot \Gamma^1_{hh}(f'_i(t))^3 + \]

\[ 2 \sum_{i=2}^{n} \Gamma^h_{1i} \ b_i f'_i(t) + \sum_{i=2}^{n} \Gamma^h_{ii} \ b_i^2 (f'_i(t))^2 + 2 \sum_{i,j=2}^{n} \Gamma^h_{ij} \ b_i b_j f'_i(t) \cdot f'_j(t) - \]

\[ 2 \sum_{i=2}^{n} \Gamma^1_{1i} \ b_i b_h \ f'_h(t) \cdot f'_i(t) - \sum_{i=2}^{n} \Gamma^1_{ii} \ b_i^2 b_h \ f'_h(t) \cdot (f'_i(t))^2 - \]

\[ 2 \sum_{i,j=2}^{n} \Gamma^1_{ij} \ b_i b_j b_h \ f'_h(t) \cdot f'_i(t) \cdot f'_j(t) = 0. \] (8)
10. Geodesics

This identity is a polynomial expression in $b_i$, $b_i^2$, $b_i^3$, $b_i b_j$ ($i \neq j$), $b_i b_h$ ($i \neq h$), $b_i^2 b_h$ ($i \neq h$), $b_i b_j b_h$ ($i \neq j$) for $i, h = 2, \ldots, n$. Since all the variables are independent, their coefficients must be zero.

Using it we get some relations from which it follows theorem
11. Geodesics

**Theorem**

Any curve of $\mathcal{X}(\mathcal{C})$ is a geodesic with respect to a connection $\nabla$ with constant coefficients $\{\Gamma^h_{ij}\}$ if and only if for $h = 2, \ldots, n$ the function $f_h(t)$ has the form

$$f_h(t) = \alpha_h e^{-2\Gamma^h_{11}t} + \beta_h \quad (\alpha_h \neq 0, \beta_h \in \mathbb{R})$$

for all $2 \leq h \leq n$.

The only components of $\nabla$ which can be different from 0 are $\Gamma^1_{11}$ and $\Gamma^h_{hh} = 2\Gamma^h_{1h}$. 

---

1. Immanuel Kant Baltic Federal University, Kaliningrad, Russia
2. Palacky University, Olomouc, Czech Republic
3. Erlangen-Nürnberg University, Erlangen, Germany

About almost geodesics curves
12. Almost geodesic curves

**Definition**

By an *almost geodesic* of an affine connection $\nabla$ we mean a piecewise $C^3$-curve $\gamma: I \to \mathbb{R}^n$ satisfying

$$\nabla_{\dot{\gamma}}(\nabla_{\dot{\gamma}}\dot{\gamma}) = \varrho \cdot \dot{\gamma} + \sigma \cdot \nabla_{\dot{\gamma}}\dot{\gamma},$$

where $\varrho, \sigma: I \to \mathbb{R}$ are continuous functions, $I \subset \mathbb{R}$ is an open interval.
13. Almost geodesic curves

Using the components of $\nabla$ the system of differential equations for almost geodesics has the form

$$
\ddot{\gamma}^h + \sum_{i,j,k=1}^{n} \left( \partial_k \Gamma^h_{ij} + \Gamma^\ell_{ij} \Gamma^h_{\ell k} \right) \dot{\gamma}^i \dot{\gamma}^j \dot{\gamma}^k + 2 \sum_{i,j=1}^{n} \Gamma^h_{ij} \ddot{\gamma}^i \dot{\gamma}^j + \sum_{i,j=1}^{n} \Gamma^h_{ij} \dot{\gamma}^i \ddot{\gamma}^j = \rho(t) \cdot \dot{\gamma}^h + \sigma(t) \cdot \left( \ddot{\gamma}^h + \sum_{i,j=1}^{n} \Gamma^h_{ij} \dot{\gamma}^i \dot{\gamma}^j \right).$$

(11)
14. Almost geodesic curves

A curve $\ell$ of $\mathfrak{X}(C)$ is an almost geodesic with respect to $\nabla$ if and only if we have

$$
\dddot{\ell}^h + \sum_{i,j,k=1}^{n} \Gamma^\ell_{ij} \Gamma^h_{\ell k} \dddot{\ell}^i \dddot{\ell}^j \dddot{\ell}^k + 2 \sum_{i,j=1}^{n} \Gamma^h_{ij} \dddot{\ell}^i \dddot{\ell}^j + \sum_{i,j=1}^{n} \Gamma^h_{ij} \dddot{\ell}^i \dddot{\ell}^j =
$$

$$
\rho(t) \cdot \dddot{\ell}^h + \sigma(t) \cdot \left( \dddot{\ell}^h + \sum_{i,j=1}^{n} \Gamma^h_{ij} \dddot{\ell}^i \dddot{\ell}^j \right).
$$

From this for $h = 1$ we obtain $\rho(t)$ and for $h = 2, \ldots, n$ after substitution of the function $\rho(t)$ we get equations (13)
15. Almost geodesic curves

\[ b_h f''''(t) + \sum_{m=1}^{n} \Gamma^h_{m1} \Gamma^m_{11} + \sum_{i=2, m=1}^{n} (\Gamma^h_{mi} \Gamma^m_{i1} + \Gamma^h_{m1} \Gamma^m_{1i} + \Gamma^h_{m1} \Gamma^m_{11}) b_i f'_i(t) + \]

\[ \sum_{i,j=2, m=1}^{n} (\Gamma^h_{mi} \Gamma^m_{ij} + \Gamma^h_{m1} \Gamma^m_{ij} + \Gamma^h_{m1} \Gamma^m_{1j}) b_i b_j f'_i(t) f'_j(t) + \]

\[ \sum_{i,j,k=2, m=1}^{n} \Gamma^h_{mi} \Gamma^m_{jk} b_i b_j b_k f'_i(t) f'_j(t) f'_k(t) + \]

\[ \sum_{i=2}^{n} (2\Gamma^h_{i1} + \Gamma^h_{1i}) b_i f''''(t) + \sum_{i,j=2}^{n} (2\Gamma^h_{ij} + \Gamma^h_{ji}) b_i b_j f''''(t) f'_j(t) - \]

\[ b_h f'_h(t) \left( \sum_{m=1}^{n} \Gamma^1_{m1} \Gamma^m_{11} + \sum_{i=2, m=1}^{n} (\Gamma^1_{mi} \Gamma^m_{i1} + \Gamma^1_{m1} \Gamma^m_{1i} + \Gamma^1_{m1} \Gamma^m_{11}) b_i f'_i(t) + \right. \]

\[ \sum_{i,j=2, m=1}^{n} (\Gamma^1_{mi} \Gamma^m_{ij} + \Gamma^1_{m1} \Gamma^m_{ij} + \Gamma^1_{m1} \Gamma^m_{1j}) b_i b_j f'_i(t) f'_j(t) + \]

\[ \sum_{i,j,k=2, m=1}^{n} \Gamma^1_{mi} \Gamma^m_{jk} b_i b_j b_k f'_i(t) f'_j(t) f'_k(t) + \]

\[ \sum_{i=2}^{n} (2\Gamma^1_{i1} + \Gamma^1_{1i}) b_i f''''(t) + \sum_{i,j=2}^{n} (2\Gamma^1_{ij} + \Gamma^1_{ji}) b_i b_j f''''(t) f'_j(t) \]
16. Almost geodesic curves

We can determine \( \sigma \) only if not all coefficients in (13) are zero, we treat first the case that for \( h \geq 2 \) one has

\[
\Gamma^h_{i1} + \Gamma^h_{1i} = \Gamma^1_{i1} + \Gamma^1_{1i}, \quad i \geq 1; \quad \Gamma^h_{ij} = \Gamma^1_{ij}, \quad i, j \geq 2. \quad (14)
\]

Since identity (13) is a polynomial expression in \( b_i, b_i b_j, b_i b_j b_k, b_h b_i b_j b_k \) \((2 \leq h, i, j, k \leq n)\) and all the variables are independent, their coefficients must be zero. Using (13) and (14) we obtain conditions, from which the next theorem follows.
17. Almost geodesic curves

Theorem

Let $C$ be a curve of the form (1) and $\nabla$ be a connection with constant coefficients $\{\Gamma^h_{ij}\}$ satisfying relations (14). Then any curve $\ell$ of $\mathfrak{X}(C)$ is almost geodesic with respect to $\nabla$ if and only if $\ell$ is represented by the functions $f_h$ having the following forms

- $f_h(t) = C_h t^2 + D_h t + E$, where $C_h, D_h, E \in \mathbb{R}$, $C_h, D_h$ not both zero and $\beta_h = 0$,
- $f_h(t) = C_h e^{\sqrt{-\beta_h} t} - D_h e^{-\sqrt{-\beta_h} t}$, where $C_h, D_h \in \mathbb{R}$, not both zero and $\beta_h < 0$,
- $f_h(t) = C_h \sin(\sqrt{\beta_h} t) - D_h \cos(\sqrt{\beta_h} t)$, where $C_h, D_h \in \mathbb{R}$, not both zero and $\beta_h > 0$,
18. Almost geodesic curves

where

\[ \beta_h = \sum_{m=1}^{n} \left( \Gamma_{mh}^h \left( \Gamma_{11}^1 + 2(\Gamma_{h1}^1 + \Gamma_{1h}^1) + 6\Gamma_{hh}^1 \right) - \Gamma_{mh}^1 \left( 2(\Gamma_{h1}^1 + \Gamma_{1h}^1) + \Gamma_{11}^1 \right) + \right. \\
\left. \Gamma_{m1}^h \left( \Gamma_{h1}^1 + \Gamma_{1h}^1 + \Gamma_{hh}^1 \right) - \Gamma_{m1}^1 \left( \Gamma_{11}^1 + \Gamma_{h1}^1 + \Gamma_{1h}^1 + 2\Gamma_{hh}^1 \right) \right). \]
Let $\alpha$ and $i_0, j_0$ such that for these indices we have

$$\Gamma_{11}^{\alpha} \neq \Gamma_{11}^1$$

or

$$\Gamma_{i_01}^{\alpha} + \Gamma_{1i_0}^{\alpha} \neq \Gamma_{i_01}^1 + \Gamma_{1i_0}^1,$$  \hspace{1cm} (15)

or

$$\Gamma_{i_0j_0}^{\alpha} \neq \Gamma_{i_0j_0}^1, \quad i_0, j_0 \geq 2.$$

In this case the coefficient of $\sigma$ is not identically zero, and we can compute $\sigma$. 
20. Almost geodesic curves

We obtain that there are the following cases:

1. $\Gamma^h_{11} = \Gamma^1_{11}$ for all $2 \leq h \leq n$ and $\Gamma^h_{i1} + \Gamma^h_{1i} = \Gamma^1_{i1} + \Gamma^1_{1i}$ for all $2 \leq i \leq n$, but there exists an $\alpha$ and $i_0, j_0$ such that $\Gamma^\alpha_{i_0j_0} \neq \Gamma^1_{i_0j_0}$.

2. $\Gamma^h_{11} = \Gamma^1_{11}$ for all $2 \leq h \leq n$, but there exists an $\alpha$ and $i_0$ such that $\Gamma^\alpha_{i_01} + \Gamma^\alpha_{1i_0} \neq \Gamma^1_{i_01} + \Gamma^1_{1i_0}$. 
20. Almost geodesic curves

We obtain that there are the following cases:

1. $\Gamma_{11}^h = \Gamma_{11}^1$ for all $2 \leq h \leq n$ and $\Gamma_{i1}^h + \Gamma_{1i}^h = \Gamma_{i1}^1 + \Gamma_{1i}^1$ for all $2 \leq i \leq n$, but there exists an $\alpha$ and $i_0, j_0$ such that $\Gamma_{i_0j_0}^\alpha \neq \Gamma_{i_0j_0}^1$.

2. $\Gamma_{11}^h = \Gamma_{11}^1$ for all $2 \leq h \leq n$, but there exists an $\alpha$ and $i_0$ such that $\Gamma_{i_01}^\alpha + \Gamma_{1i_0}^\alpha \neq \Gamma_{i_01}^1 + \Gamma_{1i_0}^1$.

3. There exists an $\alpha$ and $i_0, j_0$ such that $\Gamma_{i_0j_0}^\alpha \neq \Gamma_{i_0j_0}^1$, but $2\Gamma_{i_01}^\alpha + \Gamma_{1i_0}^1 = 0$. 

*Immanuel Kant Baltic Federal University, Kaliningrad, Russia
*Palacky University, Olomouc, Czech Republic
*Erlangen-Nürnberg University, Erlangen, Germany

About almost geodesics curves
20. Almost geodesic curves

We obtain that there are the following cases:

1. $\Gamma^h_{11} = \Gamma^1_{11}$ for all $2 \leq h \leq n$ and $\Gamma^h_{i1} + \Gamma^h_{1i} = \Gamma^1_{i1} + \Gamma^1_{1i}$ for all $2 \leq i \leq n$, but there exists an $\alpha$ and $i_0, j_0$ such that $\Gamma^\alpha_{i_0j_0} \neq \Gamma^1_{i_0j_0}$.

2. $\Gamma^h_{11} = \Gamma^1_{11}$ for all $2 \leq h \leq n$, but there exists an $\alpha$ and $i_0$ such that $\Gamma^\alpha_{i_01} + \Gamma^\alpha_{1i_0} \neq \Gamma^1_{i_01} + \Gamma^1_{1i_0}$.

3. There exists an $\alpha$ and $i_0, j_0$ such that $\Gamma^\alpha_{i_0j_0} \neq \Gamma^1_{i_0j_0}$, but $2\Gamma^\alpha_{i_01} + \Gamma^1_{1i_0} = 0$.

4. There exists an $\alpha$ and $i_0, j_0$ such that $\Gamma^\alpha_{i_0j_0} \neq \Gamma^1_{i_0j_0}$ and $2\Gamma^\alpha_{i_01} + \Gamma^1_{1i_0} \neq 0$. 

About almost geodesics curves
20. Almost geodesic curves

We obtain that there are the following cases:

1. $\Gamma_{11}^h = \Gamma_{11}^1$ for all $2 \leq h \leq n$ and $\Gamma_{i1}^h + \Gamma_{1i}^h = \Gamma_{i1}^1 + \Gamma_{1i}^1$ for all $2 \leq i \leq n$, but there exists an $\alpha$ and $i_0, j_0$ such that $\Gamma_{i_0j_0}^\alpha \neq \Gamma_{i_0j_0}^1$.

2. $\Gamma_{11}^h = \Gamma_{11}^1$ for all $2 \leq h \leq n$, but there exists an $\alpha$ and $i_0$ such that $\Gamma_{i_01}^\alpha + \Gamma_{1i_0}^\alpha \neq \Gamma_{i_01}^1 + \Gamma_{1i_0}^1$.

3. There exists an $\alpha$ and $i_0, j_0$ such that $\Gamma_{i_0j_0}^\alpha \neq \Gamma_{i_0j_0}^1$, but $2\Gamma_{i_01}^\alpha + \Gamma_{1i_0}^1 = 0$.

4. There exists an $\alpha$ and $i_0, j_0$ such that $\Gamma_{i_0j_0}^\alpha \neq \Gamma_{i_0j_0}^1$, and $2\Gamma_{i_01}^\alpha + \Gamma_{1i_0}^1 \neq 0$. 
21. Almost geodesic curves

Here we consider the case, when

\[ \Gamma^h_{11} = \Gamma^1_{11} \text{ for all } 2 \leq h \leq n \]

and

\[ \Gamma^h_{i1} + \Gamma^h_{1i} = \Gamma^1_{i1} + \Gamma^1_{1i} \text{ for all } 2 \leq i \leq n, \]

but there exists an \( \alpha \) and \( i_0, j_0 \) such that

\[ \Gamma^\alpha_{i_0j_0} \neq \Gamma^1_{i_0j_0}. \]
22. Almost geodesic curves

Theorem

Let $C$ be a curve of the form (1) and $\nabla$ be a connection with constant coefficients $\{\Gamma^h_{ij}\}$ satisfying relations (16), (17). Then any curve $\ell$ of $\mathfrak{X}(C)$ is almost geodesic with respect to $\nabla$ if and only if $\ell$ is represented by the functions $f_h$, $f_\alpha$, $f_{i0}$ having the following forms
23. Almost geodesic curves

\( f_h(t) \)

- \( f_h(t) = \hat{C}_h e^{\lambda_1^h t} + \hat{D}_h e^{\lambda_2^h t} \), where \( \hat{C}_h, \hat{D}_h \in \mathbb{R} \), \( \hat{C}_h, \hat{D}_h \) are not both zero and \( a_h^2 - 4c_h > 0 \),
- \( f_h(t) = (\tilde{C}_h t + \tilde{D}_h) e^{-\frac{a_h}{2} t} \), where \( \tilde{C}_h, \tilde{D}_h \in \mathbb{R} \), \( \tilde{C}_h, \tilde{D}_h \) are not both zero and \( a_h^2 - 4c_h = 0 \),
- \( f_h(t) = e^{-\frac{a_h}{2} t} \left( \bar{C}_h \cos \frac{\sqrt{a_h^2 - 4c_h}}{2} t + \bar{D}_h \sin \frac{\sqrt{a_h^2 - 4c_h}}{2} t \right) \), where \( \bar{C}_h, \bar{D}_h \in \mathbb{R} \), \( \bar{C}_h, \bar{D}_h \) are not both zero and \( a_h^2 - 4c_h < 0 \)

with

\[
\begin{align*}
a_h &= 2\Gamma_{h1}^h + \Gamma_{1h}^h, \quad c_h = S_{h11h} - T_{1111}, \\
\lambda_1^h &= \frac{-a_h - \sqrt{a_h^2 - 4c_h}}{2}, \quad \lambda_2^h = \frac{-a_h + \sqrt{a_h^2 - 4c_h}}{2};
\end{align*}
\]

### $f_\alpha(t)$

- $f_\alpha(t) = C_\alpha t^2 + D_\alpha t + E$, where $C_\alpha, D_\alpha, E \in \mathbb{R}$, $C_\alpha, D_\alpha$ are not both zero and $\gamma_\alpha = 0$,
- $f_\alpha(t) = \hat{C}_\alpha e^{\sqrt{-\gamma_\alpha} t} - \hat{D}_\alpha e^{-\sqrt{-\gamma_\alpha} t}$, where $\hat{C}_\alpha, \hat{D}_\alpha \in \mathbb{R}$, $\hat{C}_\alpha, \hat{D}_\alpha$ are not both zero and $\gamma_\alpha < 0$,
- $f_\alpha(t) = \hat{C}_\alpha \sin(\sqrt{\gamma_\alpha} t) - \hat{D}_\alpha \cos(\sqrt{\gamma_\alpha} t)$, where $\hat{C}_\alpha, \hat{D}_\alpha \in \mathbb{R}$, $\hat{C}_\alpha, \hat{D}_\alpha$ are not both zero and $\gamma_\alpha > 0$

with

$$\gamma_\alpha = \frac{(\Gamma^\alpha_{i_0j_0} - \Gamma^1_{i_0j_0})(T_{h\alpha ij} + T_{h\alpha ji} + T_{hi\alpha j} + T_{hij\alpha} + T_{hj\alpha i} + T_{hji\alpha})}{\Gamma^1_{ij} + \Gamma^1_{ji} - \Gamma^h_{ij} - \Gamma^h_{ji}} - T_{1111};$$
25. Almost geodesic curves

- \( f_{i_0}(t) = \hat{C}_{i_0} e^{\lambda_{i_0}^0 t} + \hat{D}_{i_0} e^{\lambda_{i_0}^0 t} \), where \( \hat{C}_{i_0}, \hat{D}_{i_0} \in \mathbb{R} \), \( \hat{C}_{i_0}, \hat{D}_{i_0} \) are not both zero and \( a_{i_0}^2 - 4c_{i_0} > 0 \),
- \( f_{i_0}(t) = (\tilde{C}_{i_0} t + \tilde{D}_{i_0}) e^{-\frac{a_{i_0}^0}{2} t} \), where \( \tilde{C}_{i_0}, \tilde{D}_{i_0} \in \mathbb{R} \), \( \tilde{C}_{i_0}, \tilde{D}_{i_0} \) are not both zero and \( a_{i_0}^2 - 4c_{i_0} = 0 \),
- \( f_{i_0}(t) = e^{-a_{i_0}^0 t/2} \left( \bar{C}_{i_0} \cos \sqrt{\frac{a_{i_0}^2 - 4c_{i_0}}{2}} t + \bar{D}_{i_0} \sin \sqrt{\frac{a_{i_0}^2 - 4c_{i_0}}{2}} t \right) \), where \( \bar{C}_{i_0}, \bar{D}_{i_0} \in \mathbb{R} \), \( \bar{C}_{i_0}, \bar{D}_{i_0} \) are not both zero and \( a_{i_0}^2 - 4c_{i_0} < 0 \)
26. Almost geodesic curves

with \( a_{i0} = 2\Gamma^i_{i01} + \Gamma^i_{1i0} \),

\[
c_{i0} = \frac{(\Gamma^i_{i0j0} - \Gamma^1_{i0j0})(T_{hi0ij} + T_{h0ij} + T_{h0i0} + T_{h0j0} + T_{h00} + T_{h00})}{\Gamma^1_{ij} + \Gamma^1_{ji} - \Gamma^h_{ij} - \Gamma^h_{ji} + S_{i0110} - T_{1111}};
\]

\[
\lambda^i_{10} = -a_{i0} - \sqrt{a^2_{i0} - 4c_{i0}} \quad \lambda^i_{20} = -a_{i0} + \sqrt{a^2_{i0} - 4c_{i0}};
\]

\[
S_{ABCD} \overset{\text{def}}{=} \sum_{m=1}^{n} \left( \Gamma^A_{mD} \Gamma^m_{BC} + \Gamma^A_{mB} (\Gamma^m_{DC} + \Gamma^m_{CD}) \right),
\]

\[
T_{ABCD} \overset{\text{def}}{=} \sum_{m=1}^{n} \Gamma^A_{mB} \Gamma^m_{CD}.
\]

We also found the relations which the components \( \{\Gamma^h_{ij}\} \) of affine connection \( \nabla \) satisfy.

---

About almost geodesic curves
Thank you for attention!

About almost geodesics curves