ABOUT ALMOST GEODESICS CURVES

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Abstract

We determine in $\mathbb{R}^n$ the form of curves $\mathcal{C}$ for which also any image under an $(n-1)$-dimensional algebraic torus is an almost geodesic with respect to an affine connections ∇ with constant coefficients and calculate explicitly the components of ∇.

1. Introduction

Geodesics are classical objects of differential geometry. They are invariants for geodesic mappings.

E. Beltrami has shown that a differentiable curve is a local geodesic with respect to an affine connection ∇ precisely if it is a solution of an Abelian differential equation with coefficients which are functions of the components of ∇.

Almost geodesics curves and mappings have been introduced in 1963 by N.S. Sinyukov as generalizations of geodesic curves and mappings.

2. Introduction

The explicit calculation of the form of curves $\mathcal{C}$ in the n-dimensional real space $\mathbb{R}^n$ which are geodesics or almost geodesics with respect to an affine connection ∇ is not achievable even in the case if the components Γ_{ij}^h of ∇ are constant. But we can do it if we suppose that with $\mathcal{C}$ also all images of $\mathcal{C}$ under a real $(n-1)$-dimensional algebraic torus are also almost geodesics.
The determination of $\mathcal{C}$ becomes an algebraic problem, namely a problem of polynomial identities.

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We consider a curve $\mathcal{C}$ homeomorphic to $\mathbb{R}$ which is a closed subset of $\mathbb{R}^n$ and has the form

$$\mathcal{C} = \Big(t, f_2(t), \dots, f_n(t)\Big), \quad t \in \mathbb{R}, \tag{1}$$

where $f_i(t)\colon\ \mathbb{R} \to \mathbb{R}$, $i = 2, \dots, n$, are three times differentiable non-constant functions.
The system

$$\mathfrak{X}(\mathcal{C}) = \Big\{\Big(t + c_1, b_2 f_2(t) + c_2, \dots, b_n f_n(t) + c_n\Big), \quad t \in \mathbb{R}\Big\}, \tag{2}$$

where $b_i \neq 0,\ c_i \in \mathbb{R}$,
is a set of imagines of $\mathcal{C}$.

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✓ If every curve of $\mathfrak{X}(\mathcal{C})$ is a geodesic with respect to an affine connection ∇ with constant coefficients Γ_{ij}^{h}, then the derivatives $f_i'(t)$ of the functions $f_i(t)$ are solutions of the first order linear ordinary differential equations.

✓ If every curve of $\mathfrak{X}(\mathcal{C})$ is an almost geodesic with respect to ∇, then the derivatives $f_i'(t)$ are solutions of harmonic oscillator equations.

✓ If $\mathfrak{X}(\mathcal{C})$ consists of Euclidean lines which are geodesic with respect to ∇, then at the most Γ_{11}^{1} may be different from 0.

In contrast to this

✓ if $\mathfrak{X}(\mathcal{C})$ consists of Euclidean lines then there is huge quantity of non-trivial connections ∇ such that the lines of $\mathfrak{X}(\mathcal{C})$ are almost geodesic with respect to ∇.

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We apply results of differential geometry only for the n-dimensional space $\mathbb{R}^n$, where global coordinates exist.
The components

$$\Gamma_{ij}^{h}, \quad h, i, j \in \{1, 2, \ldots, n\},$$

of any affine connection ∇ can be written in unique way in these coordinates.

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Let

$$\ell = \Big(t + c_1, b_2\, f_2(t) + c_2, \dots, b_n\, f_n(t) + c_n\Big), \;\; t \in \mathbb{R}, \qquad (3)$$

be a curve of $\mathfrak{X}(\mathcal{C})$.
Then

$$\begin{aligned} \dot{\ell} &= \Big(1, b_2 f_2'(t), \dots, b_n\, f_n'(t)\Big), \\ \ddot{\ell} &= \Big(0, b_2 f_2''(t), \dots, b_n\, f_n''(t)\Big). \end{aligned} \qquad (4)$$

7. Geodesics

Definition

By a *geodesic* of ∇ we mean a piecewise C^2-curve $\gamma\colon I \to \mathbb{R}^n$ satisfying

$$\nabla_{\dot\gamma}\dot\gamma = \varrho \cdot \dot\gamma, \tag{5}$$

where $\varrho\colon I \to \mathbb{R}$ is a continuous function, $I \subset \mathbb{R}$ is an open interval.

Using the components of ∇ the system of differential equations for geodesics has the form

$$\ddot\gamma^h + \sum_{i,j=1}^{n} \Gamma^h_{ij}\dot\gamma^i\dot\gamma^j = \varrho(t)\dot\gamma^h, \qquad h \in \{1, 2, \dots, n\}, \tag{6}$$

where Γ^h_{ij} are constant coefficients of an affine connections ∇.

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8. Geodesics

According to the (6) a curve ℓ of $\mathfrak{X}(\mathcal{C})$ is a geodesic with respect to ∇ if and only if

$$\ddot{\ell}_h + \sum_{i,j=1}^{n} \Gamma_{ij}^{h} \dot{\ell}_i \dot{\ell}_j = \varrho(t)\ \dot{\ell}_h, \quad h = 1, \dots, n. \tag{7}$$

From this for $h = 1$ we obtain $\varrho(t)$ and for $h = 2, \dots, n$ after substitution of the function $\varrho(t)$ we get equations (8)

9. Geodesics

$$\Gamma^h_{11}+b_h\cdot\left(f_h''(t)+2\Gamma^h_{1h}\cdot f_h'(t)\right)+{b_h}^2\cdot\left(\Gamma^h_{hh}-2\Gamma^1_{1h}\right)(f_h'(t))^2-{b_h}^3\cdot\Gamma^1_{hh}(f_i'(t))^3+$$

$$2\sum_{\substack{i=2\\ i\neq h}}^{n}\Gamma^h_{1i}\ b_i f_i'(t)+\sum_{\substack{i=2\\ i\neq h}}^{n}\Gamma^h_{ii}\ {b_i}^2(f_i'(t))^2+2\sum_{\substack{i,j=2\\ i\neq j\neq h}}^{n}\Gamma^h_{ij}\ b_i b_j f_i'(t)\cdot f_j'(t)-$$

$$2\sum_{\substack{i=2\\ i\neq h}}^{n}\Gamma^1_{1i}\ b_i b_h\ f_h'(t)\cdot f_i'(t)-\sum_{\substack{i=2\\ i\neq h}}^{n}\Gamma^1_{ii}\ {b_i}^2 b_h\ f_h'(t)\cdot(f_i'(t))^2-$$

$$2\sum_{\substack{i,j=2\\ i\neq j\neq h}}^{n}\Gamma^1_{ij}\ b_i b_j b_h\ f_h'(t)\cdot f_i'(t)\cdot f_j'(t)=0. \qquad (8)$$

10. Geodesics

This identity is a polynomial expression in b_i, b_i^2, b_i^3, $b_i b_j$ $(i \neq j)$, $b_i b_h$ $(i \neq h)$, $b_i^2 b_h$ $(i \neq h)$, $b_i b_j b_h$ $(i \neq j)$ for $i, h = 2, \dots, n$.
Since all the variables are independent, their coefficients must be zero.
Using it we get some relations from which it follows theorem

11. Geodesics

Theorem

Any curve of $\mathfrak{X}(\mathcal{C})$ is a geodesic with respect to a connection ∇ with constant coefficients $\{\Gamma^h_{ij}\}$ if and only if for $h = 2, \dots, n$ the function $f_h(t)$ has the form

$$f_h(t) = \alpha_h e^{-2\Gamma^h_{1h} \cdot t} + \beta_h \quad (\alpha_h \neq 0, \beta_h \in \mathbb{R}) \tag{9}$$

for all $2 \leq h \leq n$.
The only components of ∇ which can be different from 0 are Γ^1_{11} and $\Gamma^h_{hh} = 2\Gamma^h_{1h}$.

12. Almost geodesic curves

Definition

By an *almost geodesic* of an affine connection ∇ we mean a piecewise C^3-curve $\gamma\colon I \to \mathbb{R}^n$ satisfying

$$\nabla_{\dot\gamma}(\nabla_{\dot\gamma}\dot\gamma) = \varrho \cdot \dot\gamma + \sigma \cdot \nabla_{\dot\gamma}\dot\gamma, \tag{10}$$

where ϱ, $\sigma\colon I \to \mathbb{R}$ are continuous functions, $I \subset \mathbb{R}$ is an open interval.

13. Almost geodesic curves

Using the components of ∇ the system of differential equations for almost geodesics has the form

$$\dddot{\gamma}^h + \sum_{i,j,k=1}^{n} \left(\partial_k \Gamma^h_{ij} + \Gamma^\ell_{ij}\Gamma^h_{\ell k}\right)\dot{\gamma}^i\dot{\gamma}^j\dot{\gamma}^k + 2\sum_{i,j=1}^{n}\Gamma^h_{ij}\ddot{\gamma}^i\dot{\gamma}^j + \sum_{i,j=1}^{n}\Gamma^h_{ij}\dot{\gamma}^i\ddot{\gamma}^j =$$

$$\varrho(t)\cdot\dot{\gamma}^h + \sigma(t)\cdot\left(\ddot{\gamma}^h + \sum_{i,j=1}^{n}\Gamma^h_{ij}\dot{\gamma}^i\dot{\gamma}^j\right). \tag{11}$$

14. Almost geodesic curves

A curve ℓ of $\mathfrak{X}(\mathcal{C})$ is an almost geodesic with respect to ∇ if and only if we have

$$\dddot{\ell}^h + \sum_{i,j,k=1}^{n} \Gamma^{\ell}_{ij}\Gamma^{h}_{\ell k}\, \dot{\ell}^i \dot{\ell}^j \dot{\ell}^k + 2\sum_{i,j=1}^{n} \Gamma^h_{ij} \ddot{\ell}^i \dot{\ell}^j + \sum_{i,j=1}^{n} \Gamma^h_{ij} \dot{\ell}^i \ddot{\ell}^j =$$

$$\varrho(t)\cdot \dot{\ell}^h + \sigma(t)\cdot \left(\ddot{\ell}^h + \sum_{i,j=1}^{n} \Gamma^h_{ij} \dot{\ell}^i \dot{\ell}^j\right). \tag{12}$$

From this for $h = 1$ we obtain $\varrho(t)$ and for $h = 2, \ldots, n$ after substitution of the function $\varrho(t)$ we get equations (13)

15. Almost geodesic curves

$$
\begin{aligned}
& b_h f_h'''(t) + \sum_{m=1}^{n} \Gamma^h_{m1}\Gamma^m_{11} + \sum_{i=2,m=1}^{n} \left(\Gamma^h_{mi}\Gamma^m_{11} + \Gamma^h_{m1}\Gamma^m_{i1} + \Gamma^h_{m1}\Gamma^m_{1i}\right) b_i \; f_i'(t) + \\
& \sum_{i,j=2,m=1}^{n} \left(\Gamma^h_{mi}\Gamma^m_{j1} + \Gamma^h_{mi}\Gamma^m_{1j} + \Gamma^h_{m1}\Gamma^m_{ij}\right) b_i b_j \; f_i'(t)\, f_j'(t) + \\
& \sum_{i,j,k=2,m=1}^{n} \Gamma^h_{mi}\Gamma^m_{jk} \; b_i b_j b_k \; f_i'(t)\, f_j'(t)\, f_k'(t) + \\
& \sum_{i=2}^{n} \left(2\Gamma^h_{i1} + \Gamma^h_{1i}\right) b_i f_i''(t) + \sum_{i,j=2}^{n} \left(2\Gamma^h_{ij} + \Gamma^h_{ji}\right) b_i b_j f_i''(t) f_j'(t) - \\
& b_h f_h'(t) \Bigg(\sum_{m=1}^{n} \Gamma^1_{m1}\Gamma^m_{11} + \sum_{i=2,m=1}^{n} \left(\Gamma^1_{mi}\Gamma^m_{11} + \Gamma^1_{m1}\Gamma^m_{i1} + \Gamma^1_{m1}\Gamma^m_{1i}\right) b_i \; f_i'(t) + \\
& \sum_{i,j=2,m=1}^{n} \left(\Gamma^1_{mi}\Gamma^m_{j1} + \Gamma^1_{mi}\Gamma^m_{1j} + \Gamma^1_{m1}\Gamma^m_{ij}\right) b_i b_j \; f_i'(t)\, f_j'(t) + \\
& \sum_{i,j,k=2,m=1}^{n} \Gamma^1_{mi}\Gamma^m_{jk} \; b_i b_j b_k \; f_i'(t)\, f_j'(t)\, f_k'(t) + \\
& \sum_{i=2}^{n} \left(2\Gamma^1_{i1} + \Gamma^1_{1i}\right) b_i f_i''(t) + \sum_{i,j=2}^{n} \left(2\Gamma^1_{ij} + \Gamma^1_{ji}\right) b_i b_j f_i''(t) f_j'(t) \Bigg) = \\
& \sigma(t)\cdot\Bigg(\Gamma^h_{11} - \Gamma^1_{11} + 2\sum_{i=2}^{n}\left(\Gamma^h_{1i} + \Gamma^h_{i1} - \Gamma^1_{1i} - \Gamma^1_{i1}\right) b_i \; f_i'(t) + \sum_{i,j=2}^{n}\left(\Gamma^h_{ij} - \Gamma^1_{ij}\right) b_i b_j \; f_i'(t)\, f_j'(t)\Bigg).
\end{aligned}
\tag{13}
$$

16. Almost geodesic curves

We can determine σ only if not all coefficients in (13) are zero, we treat first the case that for $h \geq 2$ one has

$$\Gamma^h_{i1} + \Gamma^h_{1i} = \Gamma^1_{i1} + \Gamma^1_{1i}, \quad i \geq 1; \qquad \Gamma^h_{ij} = \Gamma^1_{ij}, \quad i, j \geq 2. \tag{14}$$

Since identity (13) is a polynomial expression in b_i, $b_i b_j$, $b_i b_j b_k$, $b_h b_i b_j b_k$ $(2 \leq h, i, j, k \leq n)$ and all the variables are independent, their coefficients must be zero.

Using (13) and (14) we obtain conditions, from which the next theorem follows.

17. Almost geodesic curves

Theorem

Let $\mathcal{C}$ be a curve of the form (1) *and ∇ be a connection with constant coefficients $\{\Gamma^h_{ij}\}$ satisfying relations* (14).
Then any curve ℓ of $\mathfrak{X}(\mathcal{C})$ is almost geodesic with respect to ∇ if and only if ℓ is represented by the functions f_h having the following forms

- $f_h(t) = C_h t^2 + D_h t + E$, *where* $C_h, D_h, E \in \mathbb{R}$, C_h, D_h *not both zero and* $\beta_h = 0$,
- $f_h(t) = C_h e^{\sqrt{-\beta_h}\,t} - D_h e^{-\sqrt{-\beta_h}\,t}$, *where* $C_h, D_h \in \mathbb{R}$, *not both zero and* $\beta_h < 0$,
- $f_h(t) = C_h \sin(\sqrt{\beta_h}\,t) - D_h \cos(\sqrt{\beta_h}\,t)$, *where* $C_h, D_h \in \mathbb{R}$, *not both zero and* $\beta_h > 0$,

18. Almost geodesic curves

where

$$\beta_h = \sum_{m=1}^{n} \left(\Gamma^h_{mh} \left(\Gamma^1_{11} + 2(\Gamma^1_{h1} + \Gamma^1_{1h}) + 6\Gamma^1_{hh} \right) - \Gamma^1_{mh} \left(2(\Gamma^1_{h1} + \Gamma^1_{1h}) + \Gamma^1_{11} \right) + \right.$$

$$\left. \Gamma^h_{m1}(\Gamma^1_{h1} + \Gamma^1_{1h} + \Gamma^1_{hh}) - \Gamma^1_{m1} \left(\Gamma^1_{11} + \Gamma^1_{h1} + \Gamma^1_{1h} + 2\Gamma^1_{hh} \right) \right).$$

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19. Almost geodesic curves

Let α and i_0, j_0 such that for these indices we have

$$\Gamma^{\alpha}_{11} \neq \Gamma^{1}_{11}$$

or

$$\Gamma^{\alpha}_{i_0 1} + \Gamma^{\alpha}_{1 i_0} \neq \Gamma^{1}_{i_0 1} + \Gamma^{1}_{1 i_0}, \tag{15}$$

or

$$\Gamma^{\alpha}_{i_0 j_0} \neq \Gamma^{1}_{i_0 j_0}, \quad i_0, j_0 \geq 2.$$

In this case the coefficient of σ is not identically zero, and we can compute σ.

20. Almost geodesic curves

We obtain that there are the following cases:

1. $\Gamma^h_{11} = \Gamma^1_{11}$ for all $2 \leq h \leq n$ and $\Gamma^h_{i1} + \Gamma^h_{1i} = \Gamma^1_{i1} + \Gamma^1_{1i}$ for all $2 \leq i \leq n$, but there exists an α and i_0, j_0 such that $\Gamma^\alpha_{i_0 j_0} \neq \Gamma^1_{i_0 j_0}$.
2. $\Gamma^h_{11} = \Gamma^1_{11}$ for all $2 \leq h \leq n$, but there exists an α and i_0 such that $\Gamma^\alpha_{i_0 1} + \Gamma^\alpha_{1 i_0} \neq \Gamma^1_{i_0 1} + \Gamma^1_{1 i_0}$.

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2. $\Gamma^h_{11} = \Gamma^1_{11}$ for all $2 \le h \le n$, but there exists an α and i_0 such that $\Gamma^\alpha_{i_0 1} + \Gamma^\alpha_{1 i_0} \neq \Gamma^1_{i_0 1} + \Gamma^1_{1 i_0}$.
3. There exists an α and i_0, j_0 such that $\Gamma^\alpha_{i_0 j_0} \neq \Gamma^1_{i_0 j_0}$, but $2\Gamma^\alpha_{i_0 1} + \Gamma^1_{1 i_0} = 0$.

20. Almost geodesic curves

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2. $\Gamma^h_{11} = \Gamma^1_{11}$ for all $2 \le h \le n$, but there exists an α and i_0 such that $\Gamma^\alpha_{i_0 1} + \Gamma^\alpha_{1 i_0} \ne \Gamma^1_{i_0 1} + \Gamma^1_{1 i_0}$.
3. There exists an α and i_0, j_0 such that $\Gamma^\alpha_{i_0 j_0} \ne \Gamma^1_{i_0 j_0}$, but $2\Gamma^\alpha_{i_0 1} + \Gamma^1_{1 i_0} = 0$.
4. There exists an α and i_0, j_0 such that $\Gamma^\alpha_{i_0 j_0} \ne \Gamma^1_{i_0 j_0}$ and $2\Gamma^\alpha_{i_0 1} + \Gamma^1_{1 i_0} \ne 0$.

20. Almost geodesic curves

We obtain that there are the following cases:

1. $\Gamma^h_{11} = \Gamma^1_{11}$ for all $2 \le h \le n$ and $\Gamma^h_{i1} + \Gamma^h_{1i} = \Gamma^1_{i1} + \Gamma^1_{1i}$ for all $2 \le i \le n$, but there exists an α and i_0, j_0 such that $\Gamma^\alpha_{i_0 j_0} \ne \Gamma^1_{i_0 j_0}$.
2. $\Gamma^h_{11} = \Gamma^1_{11}$ for all $2 \le h \le n$, but there exists an α and i_0 such that $\Gamma^\alpha_{i_0 1} + \Gamma^\alpha_{1 i_0} \ne \Gamma^1_{i_0 1} + \Gamma^1_{1 i_0}$.
3. There exists an α and i_0, j_0 such that $\Gamma^\alpha_{i_0 j_0} \ne \Gamma^1_{i_0 j_0}$, but $2\Gamma^\alpha_{i_0 1} + \Gamma^1_{1 i_0} = 0$.
4. There exists an α and i_0, j_0 such that $\Gamma^\alpha_{i_0 j_0} \ne \Gamma^1_{i_0 j_0}$ and $2\Gamma^\alpha_{i_0 1} + \Gamma^1_{1 i_0} \ne 0$.

21. Almost geodesic curves

Here we consider the case, when

$$\begin{gathered}\Gamma^h_{11} = \Gamma^1_{11} \text{ for all } 2 \leq h \leq n \\ \text{and} \\ \Gamma^h_{i1} + \Gamma^h_{1i} = \Gamma^1_{i1} + \Gamma^1_{1i} \text{ for all } 2 \leq i \leq n,\end{gathered} \tag{16}$$

but there exists an α and i_0, j_0 such that

$$\Gamma^\alpha_{i_0 j_0} \neq \Gamma^1_{i_0 j_0}. \tag{17}$$

22. Almost geodesic curves

Theorem

Let $\mathcal{C}$ be a curve of the form (1) and ∇ be a connection with constant coefficients $\{\Gamma^h_{ij}\}$ satisfying relations (16), (17). Then any curve ℓ of $\mathfrak{X}(\mathcal{C})$ is almost geodesic with respect to ∇ if and only if ℓ is represented by the functions f_h, f_α, f_{i_0} having the following forms

23. Almost geodesic curves

$f_h(t)$

- $f_h(t) = \hat{C}_h e^{\lambda_1^h t} + \hat{D}_h e^{\lambda_2^h t}$, where $\hat{C}_h, \hat{D}_h \in \mathbb{R}$, $\hat{C}_h, \hat{D}_h$ are not both zero and $a_h^2 - 4c_h > 0$,
- $f_h(t) = (\tilde{C}_h t + \tilde{D}_h) e^{\frac{-a_h}{2} t}$, where $\tilde{C}_h, \tilde{D}_h \in \mathbb{R}$, $\tilde{C}_h, \tilde{D}_h$ are not both zero and $a_h^2 - 4c_h = 0$,
- $f_h(t) = e^{-a_h t/2}\left(\bar{C}_h cos\frac{\sqrt{a_h^2-4c_h}}{2}t + \bar{D}_h sin\frac{\sqrt{a_h^2-4c_h}}{2}t\right)$, where $\bar{C}_h, \bar{D}_h \in \mathbb{R}$, $\bar{C}_h, \bar{D}_h$ are not both zero and $a_h^2 - 4c_h < 0$

with

$$a_h = 2\Gamma_{h1}^h + \Gamma_{1h}^h, \quad c_h = S_{h11h} - T_{1111},$$

$$\lambda_1^h = \frac{-a_h - \sqrt{a_h^2 - 4c_h}}{2}, \quad \lambda_2^h = \frac{-a_h + \sqrt{a_h^2 - 4c_h}}{2};$$

24. Almost geodesic curves

$f_\alpha(t)$

- $f_\alpha(t) = C_\alpha t^2 + D_\alpha t + E$, where $C_\alpha, D_\alpha, E \in \mathbb{R}$, C_α, D_α are not both zero and $\gamma_\alpha = 0$,
- $f_\alpha(t) = \hat{C}_\alpha e^{\sqrt{-\gamma_\alpha}\, t} - \hat{D}_\alpha e^{-\sqrt{-\gamma_\alpha}\, t}$, where $\hat{C}_\alpha, \hat{D}_\alpha \in \mathbb{R}$, $\hat{C}_\alpha, \hat{D}_\alpha$ are not both zero and $\gamma_\alpha < 0$,
- $f_\alpha(t) = \hat{C}_\alpha \sin(\sqrt{\gamma_\alpha}\, t) - \hat{D}_\alpha \cos(\sqrt{\gamma_\alpha}\, t)$, where $\hat{C}_\alpha, \hat{D}_\alpha \in \mathbb{R}$, $\hat{C}_\alpha, \hat{D}_\alpha$ are not both zero and $\gamma_\alpha > 0$

with

$$\gamma_\alpha = \frac{(\Gamma^\alpha_{i_0 j_0} - \Gamma^1_{i_0 j_0})(T_{h\alpha ij} + T_{h\alpha ji} + T_{hi\alpha j} + T_{hij\alpha} + T_{hj\alpha i} + T_{hji\alpha})}{\Gamma^1_{ij} + \Gamma^1_{ji} - \Gamma^h_{ij} - \Gamma^h_{ji}} - T_{1111};$$

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$f_{i_0}(t)$

- $f_{i_0}(t) = \hat{C}_{i_0} e^{\lambda_1^{i_0} t} + \hat{D}_{i_0} e^{\lambda_2^{i_0} t}$, where $\hat{C}_{i_0}, \hat{D}_{i_0} \in \mathbb{R}$, $\hat{C}_{i_0}, \hat{D}_{i_0}$ are not both zero and $a_{i_0}^2 - 4c_{i_0} > 0$,
- $f_{i_0}(t) = (\tilde{C}_{i_0} t + \tilde{D}_{i_0}) e^{\frac{-a_{i_0}}{2} t}$, where $\tilde{C}_{i_0}, \tilde{D}_{i_0} \in \mathbb{R}$, $\tilde{C}_{i_0}, \tilde{D}_{i_0}$ are not both zero and $a_{i_0}^2 - 4c_{i_0} = 0$,
- $f_{i_0}(t) = e^{-a_{i_0} t/2} \left(\bar{C}_{i_0} cos \frac{\sqrt{a_{i_0}^2 - 4c_{i_0}}}{2} t + \bar{D}_{i_0} sin \frac{\sqrt{a_{i_0}^2 - 4c_{i_0}}}{2} t \right)$, where $\bar{C}_{i_0}, \bar{D}_{i_0} \in \mathbb{R}$, $\bar{C}_{i_0}, \bar{D}_{i_0}$ are not both zero and $a_{i_0}^2 - 4c_{i_0} < 0$

26. Almost geodesic curves

with $a_{i_0} = 2\Gamma^{i_0}_{i_0 1} + \Gamma^{i_0}_{1 i_0}$,

$$c_{i_0} = \frac{(\Gamma^{i_0}_{i_0 j_0} - \Gamma^{1}_{i_0 j_0})(T_{h i_0 ij} + T_{h i_0 ji} + T_{h i i_0 j} + T_{h i j i_0} + T_{h j i_0 i} + T_{h j i i_0})}{\Gamma^1_{ij} + \Gamma^1_{ji} - \Gamma^h_{ij} - \Gamma^h_{ji}} + S_{i_0 1 1 i_0} - T_{1111},$$

$$\lambda^{i_0}_1 = \frac{-a_{i_0} - \sqrt{a^2_{i_0} - 4c_{i_0}}}{2}, \quad \lambda^{i_0}_2 = \frac{-a_{i_0} + \sqrt{a^2_{i_0} - 4c_{i_0}}}{2};$$

$$S_{ABCD} \overset{def}{=} \sum_{m=1}^{n} \Big(\Gamma^A_{mD}\Gamma^m_{BC} + \Gamma^A_{mB}(\Gamma^m_{DC} + \Gamma^m_{CD}) \Big),$$

$$T_{ABCD} \overset{def}{=} \sum_{m=1}^{n} \Gamma^A_{mB}\Gamma^m_{CD}.$$

We also found the relations which the components $\{\Gamma^h_{ij}\}$ of affine connection ∇ satisfy.

Thank you for attention !

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