The Hankel transform of aerated sequences

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Abstract

This paper provides the connection between the Hankel transform and aerating transforms of a given integer sequence. Two different aerating transforms are introduced and closed-form expressions are derived for the Hankel transform of such aerated sequences. Combinations of both aerating and Hankel transform are also considered. Our results are general and can be applied to a wide class of integer sequences. As an application, we use our tools on the sequence of shifted Catalan numbers $(C_{n+t})_{n\in\mathbb{N}_0}$. For that purpose, we need to evaluate the Hankel and Hankel-like determinants based on the Catalan numbers. Our approach is based on the results of Gessel and Viennot [7] and more recent results of Krattenthaler [10]. We generalize a sequence obtained by the series reversion of $Q(x) = \frac{x}{1+\alpha x+\beta x^2}$ (studied in our previous paper [2]), and provide the Hankel transform evaluation of that sequence and its shifted sequences.

Key words: Hankel determinant, Catalan numbers, aerating transform, series reversion.

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1 Introduction

The Hankel transform of a given sequence $a = (a_n)_{n\in\mathbb{N}_0}$ is defined as the sequence $h = (h_n)_{n\in\mathbb{N}_0}$ of Hankel determinants, i.e.

$$h_n = \det\left([a_{i+j}]_{0\le i,j\le n}\right), \quad (n \in \mathbb{N}_0) \tag{1}$$

and denoted by $h = \mathcal{H}(a)$. We also denote by $\mathbf{H}_a$ the (infinite) *Hankel matrix* of the sequence a defined by

$$\mathbf{H}_a = [a_{i+j}]_{i,j\in\mathbb{N}_0}.$$

The term "Hankel transform" was first introduced by Layman [11] in 2001. However, many Hankel determinant evaluations were obtained much earlier, mostly due to their important combinatorial properties (see for example [3, 8, 15, 16]).

Papers [3, 5, 12] use a method based on orthogonal polynomials (or continued fractions) to provide a Hankel transform evaluation of different sequences. Such methods are also used in our recently published paper [2] where we evaluated the Hankel transform of a series reversion of the function $Q(x) = \frac{x}{1+\alpha x+\beta x^2}$, as well as of the corresponding shifted sequences.

In this paper, we consider the relationship between the Hankel transform and two different types of aerating transforms. We will give closed-form expressions establishing a relationship between the Hankel transform of the original and aerated sequence. We also study a transformation which is a mixture of the aerating and the Hankel transform (called the Hankel-aerating transform), and give a closed-form expression for its evaluation. All established results are general and can be efficiently applied for different types of sequences.

We apply these results to the sequence of shifted Catalan numbers $(C_{n+t})_{n\in\mathbb{N}_0}$. For that purpose, we need to evaluate several Hankel and Hankel-like determinants based on Catalan numbers $(C_n)_{n\in\mathbb{N}_0}$. Our approach is based on the well-known result by Gessel and Viennot [7] and recent results of Krattenthaler [10]. Those evaluations include the Hankel transform of $(\alpha^2 C_n - \beta C_{n+1})_{n\in\mathbb{N}_0}$ and of $(\alpha^2 C_{n+1} - \beta C_{n+2})_{n\in\mathbb{N}_0}$, which provides a direct generalization of results by Cvetković, Rajković and Ivković [5].

The other application is to the generalization of results obtained in our previous paper [2]. We generalize the sequence obtained from the series reversion of $\frac{x}{1+\alpha x+\beta x^2}$ as well as the corresponding shifted sequences, and provide their Hankel transform evaluations.

2 The Hankel transform of aerated sequences

By the term *aerated sequence*, we will understand a sequence of the form $(c_0, 0, c_1, 0, c_2, 0, \dots)$, where $(c_n)_{n\in\mathbb{N}_0}$ is a given sequence. We are led to define the following *aerating transform*.

Definition 2.1. *For a given sequence $c = (c_n)_{n\in\mathbb{N}_0}$, we define its aerating transformation $p = \mathcal{A}(c)$ by*

$$p_n = \begin{cases} c_{n/2}, & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases}.$$

In other words, if $p = \mathcal{A}(c)$ then $p = (c_0, 0, c_1, 0, c_2, 0, c_3, 0, \dots)$.

The next theorem shows the connection between the Hankel transform of a given sequence c and its aerated sequence $p = \mathcal{A}(c)$. It is based on the well-known formula for the determinant of a "checkerboard" matrix.

Theorem 2.1. *Let $g = \mathcal{H}(p)$ and $h = \mathcal{H}(c)$ where $p = \mathcal{A}(c)$ is the aerated sequence of c. Then*

$$\det[p_{i+j}]_{0\le i,j\le n} = \det[c_{i+j}]_{0\le i,j\le \lfloor \frac{n}{2}\rfloor} \cdot \det[c_{i+j+1}]_{0\le i,j\le \lfloor \frac{n-1}{2}\rfloor}.$$

In terms of Hankel transforms, this last equality can be written as $g_n = h_{\lfloor \frac{n}{2}\rfloor} h^*_{\lfloor \frac{n-1}{2}\rfloor}$, where h^* is the Hankel transform of the shifted sequence $c^* = (c_{n+1})_{n\in\mathbb{N}_0}$, that is, $h^* = \mathcal{H}(c^*)$.

In a similar way, we define another aerating-type transformation which is called the α-aerating transform and denoted by $\mathcal{A}(c;\alpha)$.

Definition 2.2. *For a given sequence $c = (c_n)_{n\in\mathbb{N}_0}$, we define its α-aerating transform $a = \mathcal{A}(c;\alpha)$ by $a_n = \alpha p_n + p_{n+1}$, where $p = \mathcal{A}(c)$. In other words, if $a = \mathcal{A}(c;\alpha)$ then $a = (\alpha c_0, c_1, \alpha c_1, c_2, \alpha c_2, c_3, \alpha c_3, \dots)$.*

From now on, we denote by $[\mathbf{A}]_{m\times m}$ a matrix formed by the first m rows and columns of the (infinite) matrix $\mathbf{A}$. We also assume that matrix indices start from 0.

If $c = (c_n)_{n\in\mathbb{N}_0}$ is a given sequence, then the Hankel transform $g = \mathcal{H}(a)$ of the α-aerated sequence $a = \mathcal{A}(c;\alpha)$ has the form

$$g_n = \det[a_{i+j}]_{0\le i,j\le n} = \det \begin{bmatrix} \alpha c_0 & c_1 & \alpha c_1 & c_2 & \cdots \\ c_1 & \alpha c_1 & c_2 & \alpha c_2 & \\ \alpha c_1 & c_2 & \alpha c_2 & c_3 & \\ c_2 & \alpha c_2 & c_3 & \alpha c_3 & \\ \vdots & & & & \ddots \end{bmatrix}_{(n+1)\times(n+1)}.$$

Its evaluation is given by the following theorem.

Theorem 2.2. *Let $g = \mathcal{H}(a)$ and $a = \mathcal{A}(c;\alpha)$. Then*

$$g_n = \det[a_{i+j}]_{0\le i,j\le n} = \begin{cases} \det[\alpha^2 c_{i+j} - c_{i+j+1}]_{0\le i,j\le k-1} \cdot \det[c_{i+j+1}]_{0\le i,j\le k-1}, & n = 2k-1 \\ \alpha \cdot \det[\alpha^2 c_{i+j+1} - c_{i+j+2}]_{0\le i,j\le k-1} \cdot \det[c_{i+j}]_{0\le i,j\le k}, & n = 2k \end{cases}. \quad (2)$$

Proof. We distinguish between two cases depending on the parity of n.

Case 1. $n = 2k-1$ is odd. We subtract α^{-1} times column $2j+1$ from column $2j$, for every $j = 0,1,\ldots,k-1$. That leads to the determinant

$$\det[a_{i+j}]_{0\le i,j\le n} = \det \begin{bmatrix} \alpha c_0 - \alpha^{-1}c_1 & c_1 & \alpha c_1 - \alpha^{-1}c_2 & c_2 & \cdots \\ 0 & \alpha c_1 & 0 & \alpha c_2 & \\ \alpha c_1 - \alpha^{-1}c_2 & c_2 & \alpha c_2 - \alpha^{-1}c_3 & c_3 & \\ 0 & \alpha c_2 & 0 & \alpha c_3 & \\ \vdots & & & & \ddots \end{bmatrix}_{(n+1)\times(n+1)}$$

By permuting rows and columns appropriately, we get the block diagonal form

$$\det[a_{i+j}]_{0\le i,j\le n} = \det \begin{bmatrix} \mathbf{A} & * \\ & \mathbf{B} \end{bmatrix} = \det \mathbf{A} \cdot \det \mathbf{B},$$

where star $(*)$ denotes the appropriate $k\times k$ matrix which does not have any influence on the determinant computation. Matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = [\alpha c_{i+j} - \alpha^{-1}c_{i+j+1}]_{0\le i,j\le k-1}, \quad \mathbf{B} = [\alpha c_{i+j+1}]_{0\le i,j\le k-1}.$$

Now since

$$\begin{aligned} \det \mathbf{A} \cdot \det \mathbf{B} &= \alpha^{-k} \det[\alpha^2 c_{i+j} - c_{i+j+1}]_{0\le i,j\le k-1} \cdot \alpha^k \det[c_{i+j+1}]_{0\le i,j\le k-1} \\ &= \det[\alpha^2 c_{i+j} - c_{i+j+1}]_{0\le i,j\le k-1} \cdot \det[c_{i+j+1}]_{0\le i,j\le k-1} \end{aligned}$$

we get the first case of expression (2).

Case 2. $n = 2k$ is even. Now by subtracting α^{-1} times column $2j$ from column $2j-1$ $(j = 1,2,\ldots,k)$, we get

$$\det[a_{i+j}]_{0\le i,j\le n} = \det \begin{bmatrix} \alpha c_0 & 0 & \alpha c_1 & 0 & \alpha c_2 & \cdots \\ c_1 & \alpha c_1 - \alpha^{-1}c_2 & c_2 & \alpha c_2 - \alpha^{-1}c_3 & c_3 & \\ \alpha c_1 & 0 & \alpha c_2 & 0 & \alpha c_3 & \\ c_2 & \alpha c_2 - \alpha^{-1}c_3 & c_3 & \alpha c_3 - \alpha^{-1}c_4 & \alpha c_3 & \\ \alpha c_2 & 0 & \alpha c_3 & 0 & \alpha c_4 & \\ \vdots & & & & & \ddots \end{bmatrix}_{(n+1)\times(n+1)}.$$

Again we permute rows and columns appropriately to get the block diagonal form

$$\det[a_{i+j}]_{0\le i,j\le n} = \det \begin{bmatrix} \mathbf{A}' & * \\ & \mathbf{B}' \end{bmatrix} = \det \mathbf{A}' \cdot \det \mathbf{B}',$$

where

$$\mathbf{A}' = [\alpha c_{i+j}]_{0\le i,j\le k}, \quad \mathbf{B}' = [\alpha c_{i+j+1} - \alpha^{-1} c_{i+j+2}]_{0\le i,j\le k-1}.$$

Now since

$$\begin{aligned} \det \mathbf{A}' \cdot \det \mathbf{B}' &= \alpha^{k+1} \det[c_{i+j}]_{0\le i,j\le k} \cdot \alpha^{-k} \det[\alpha^2 c_{i+j+1} - c_{i+j+2}]_{0\le i,j\le k-1} \\ &= \alpha \cdot \det[c_{i+j}]_{0\le i,j\le k} \cdot \det[\alpha^2 c_{i+j+1} - c_{i+j+2}]_{0\le i,j\le k-1} \end{aligned}$$

we obtain the second part of (2). □

Now consider the matrix $\tilde{\mathbf{H}}_c$, formed by adding an additional row and column to the Hankel matrix $\mathbf{H}_a$ of the α-aerated sequence $a = \mathcal{A}(c, \alpha)$:

$$\tilde{\mathbf{H}}_c = \begin{bmatrix} 0 & p^T \\ p & \mathbf{H}_a \end{bmatrix} = \begin{bmatrix} 0 & c_0 & 0 & c_1 & 0 & \cdots \\ c_0 & \alpha c_0 & c_1 & \alpha c_1 & c_2 & \\ 0 & c_1 & \alpha c_1 & c_2 & \alpha c_2 & \\ c_1 & \alpha c_1 & c_2 & \alpha c_2 & c_3 & \\ 0 & c_2 & \alpha c_2 & c_3 & \alpha c_3 & \\ \vdots & & & & & \ddots \end{bmatrix}.$$

Here, we have used $p = \mathcal{A}(c)$ and we treat a sequence as an infinite column vector.

Definition 2.3. *The Hankel-aerating transform of the sequence $c = (c_n)_{n\in\mathbb{N}_0}$ is the sequence $\left(\tilde{h}_n\right)_{n\in\mathbb{N}_0}$ defined by*

$$\tilde{h}_n = \det[\tilde{\mathbf{H}}_c]_{(n+1)\times(n+1)}.$$

The following theorem provides the evaluation of the Hankel-aerating transform. Note that we use the notation $\chi(P) = 1$ if P is true and $\chi(P) = 0$ otherwise.

Theorem 2.3. *If $p = \mathcal{A}(c)$ and $a = \mathcal{A}(c;\alpha)$, then*

$$\tilde{h}_n = \det[\tilde{\mathbf{H}}_c]_{(n+1)\times(n+1)} = P_1 \cdot P_2$$

where

$$P_1 = \begin{cases} \det[c_{i+j}]_{0\le i,j\le k-1}, & n = 2k-1 \\ \det[c_{i+j+1}]_{0\le i,j\le k-1}, & n = 2k \end{cases}$$

$$P_2 = \begin{cases} \displaystyle\sum_{l=0}^{k-1} (-1)^{k+l} \left(\sum_{h=0}^{l} \alpha^{2h} c_{l-h} \right) \det[c_{i+j+\chi(j\ge l)+1}]_{0\le i,j\le k-2}, & n = 2k-1 \\ \displaystyle\sum_{l=1}^{k} (-1)^{k+l+1} \left(\sum_{h=0}^{l-1} \alpha^{2h+1} c_{l-1-h} \right) \det[c_{i+j+\chi(j\ge l)}]_{0\le i,j\le k-1}, & n = 2k \end{cases}$$

Proof. We again distinguish between two cases depending on the parity of n.

Case 1. $n = 2k-1$ is odd. By subtracting α times column $2j$ from column $2j+1$ ($j = 0,1,\dots,k-1$), we get

$$\tilde{h}_n = \det \begin{bmatrix} 0 & c_0 & 0 & c_1 & 0 & \cdots \\ c_0 & 0 & c_1 & 0 & c_2 & \\ 0 & c_1 & \alpha c_1 & c_2 - \alpha^2 c_1 & \alpha c_2 & \\ c_1 & 0 & c_2 & 0 & c_3 & \\ 0 & c_2 & \alpha c_2 & c_3 - \alpha^2 c_2 & \alpha c_3 & \\ \vdots & & & & & \ddots \end{bmatrix}_{(n+1)\times(n+1)}.$$

Permuting rows and columns in the previous determinant yields

$$\tilde{h}_n = (-1)^k \det \begin{bmatrix} \mathbf{A} & * \\ & \mathbf{B} \end{bmatrix} = (-1)^k \det \mathbf{A} \cdot \det \mathbf{B} \tag{3}$$

where the matrices $\mathbf{A}$ and $\mathbf{B}$ are equal to

$$\mathbf{A} = \det \begin{bmatrix} c_0 & c_1 & \cdots & c_{k-1} \\ c_1 & c_2 - \alpha^2 c_1 & & c_{k-2} - \alpha^2 c_{k-1} \\ \vdots & & \ddots & \\ c_{k-1} & c_{k-2} - \alpha^2 c_{k-1} & & c_{2k-2} - \alpha^2 c_{2k-3} \end{bmatrix}, \quad \mathbf{B} = [c_{i+j}]_{0\le i,j\le k-1}.$$

By successively adding α^2 times column j to column $j+1$ of the matrix $\mathbf{A}$ ($j = 0,1,\dots,k-2$) we obtain the following determinant

$$\det \mathbf{A} = \det \begin{bmatrix} c_0 & \alpha^2 c_0 + c_1 & \alpha^4 c_0 + \alpha^2 c_1 + c_2 & \cdots \\ c_1 & c_2 & c_3 & \\ c_2 & c_3 & c_4 & \\ \vdots & & & \ddots \end{bmatrix}_{k\times k}.$$

Expansion along the first row yields

$$\det \mathbf{A} = \sum_{l=0}^{k-1} (-1)^l \left(\sum_{h=0}^{l} \alpha^{2h} c_{l-h} \right) \det[c_{i+j+\chi(j\ge l)+1}]_{0\le i,j\le k-2}. \tag{4}$$

Now combining (3) and (4), we obtain the statement of the theorem for $n = 2k-1$.

Case 2. $n = 2k$ is even. By subtracting α times column $2j-1$ from column $2j$ ($j = 1,2,\dots,k$), we obtain

$$h_n = \det \begin{bmatrix} 0 & c_0 & -\alpha c_0 & c_1 & 0 & \cdots \\ c_0 & \alpha c_0 & c_1 - \alpha^2 c_0 & \alpha c_1 & c_2 - \alpha^2 c_1 & \\ 0 & c_1 & 0 & c_2 & 0 & \\ c_1 & \alpha c_1 & c_2 - \alpha^2 c_1 & \alpha c_2 & c_3 - \alpha^2 c_2 & \\ 0 & c_2 & 0 & c_3 & 0 & \\ \vdots & & & & & \ddots \end{bmatrix}_{(n+1)\times(n+1)}.$$

By permuting rows and columns appropriately, in the previous determinant we obtain

$$h_n = (-1)^k \det \begin{bmatrix} \mathbf{A}' & * \\ & \mathbf{B}' \end{bmatrix} = (-1)^k \det \mathbf{A}' \cdot \det \mathbf{B}' \tag{5}$$

where the matrices $\mathbf{A}'$ and $\mathbf{B}'$ are equal to

$$\mathbf{A}' = \det \begin{bmatrix} 0 & -\alpha c_0 & \cdots & -\alpha c_{k-1} \\ c_0 & c_1 - \alpha^2 c_0 & & c_k - \alpha^2 c_{k-1} \\ \vdots & & \ddots & \\ c_{k-1} & c_k - \alpha^2 c_{k-1} & & c_{2k-1} - \alpha^2 c_{2k-2} \end{bmatrix}, \quad \mathbf{B}' = [c_{i+j+1}]_{0\le i,j\le k-1}.$$

Again, by adding α^2 times column j to column $j+1$ ($j = 0, 1, \ldots, k-1$), we obtain the determinant

$$\det \mathbf{A}' = \det \begin{bmatrix} 0 & -\alpha c_0 & -\alpha^3 c_0 - \alpha c_1 & -\alpha^5 c_0 - \alpha^3 c_1 - \alpha c_2 & \cdots \\ c_0 & c_1 & c_2 & c_3 & \\ c_1 & c_2 & c_3 & c_4 & \\ c_2 & c_3 & c_4 & c_5 & \\ \vdots & & & & \ddots \end{bmatrix}_{(k+1)\times(k+1)}.$$

which can be expanded along the first row in the following way:

$$\det \mathbf{A}' = \sum_{l=1}^{k} (-1)^{l+1} \left(\sum_{h=0}^{l-1} \alpha^{2h+1} c_{l-1-h} \right) \det[c_{i+j+\chi(j\ge l)}]_{0\le i,j\le k-1}. \tag{6}$$

Now combining (5) and (6), we obtain the statement of the theorem for $n = 2k$. □

3 Hankel and Hankel-like determinant evaluations based on Catalan numbers

In this section, we present the evaluation of determinants based on the sequence of Catalan numbers $C = (C_n)_{n\in\mathbb{N}_0}$, which will be useful in the rest of the paper. Our main tool is the following theorem proven by Gessel and Viennot in [7] and restated by Krattenthaler in [10] (Theorem 3):

Theorem 3.1. [7, 10] *Let n be a positive integer and $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ non-negative integers. Then*

$$\det[C_{\alpha_i+j}]_{0\le i,j\le n-1} = \prod_{0\le i<j\le n-1} (\alpha_j - \alpha_i) \prod_{i=0}^{n-1} \frac{(i+n)!(2\alpha_i)!}{(2i)!\alpha_i!(\alpha_i+n)!}. \tag{7}$$

Corollary 3.2. *For every $t \in \mathbb{N}_0$, the Hankel transform of the sequence $(C_{n+t})_{n\in\mathbb{N}_0}$ is given by*

$$\det[C_{i+j+t}]_{0\le i,j\le n-1} = \prod_{p=0}^{t-1} \frac{p!(2n+2p)!}{(2p)!(2n+p)!}$$

The next two corollaries follow directly from Theorem 3.1, by taking $\alpha_i = i + \chi(i \ge l)$ and $\alpha_i = i + \chi(i \ge l) + t$ respectively. Recall that $\chi(P) = 1$ if P is true and $\chi(P) = 0$ otherwise.

Corollary 3.3. *For every $l = 0, 1, \ldots, n-1$ we have*

$$\det[C_{i+j+\chi(j\ge l)}]_{0\le i,j\le n-1} = \binom{l+n}{2l}.$$

Corollary 3.4. *For every $l = 0, 1, \ldots, n-1$ and every $t \in \mathbb{N}_0$ we have*

$$\det[C_{i+j+\chi(j\geq l)+t}]_{0\leq i,j\leq n-1} = \binom{l+n}{2l} \prod_{p=0}^{t-1} \frac{p!(2n+2p+1)!(p+n+l+1)}{(2p)!(2n+p+1)!(2p+2l+1)}$$

The following lemma provides a straightforward generalization of Lemma 4 in [10].

Lemma 3.5. *If $\mathbf{A} = [a_{i,j}]_{i,j\in\mathbb{N}_0}$ is a given matrix, then*

$$\det[Aa_{i,j} + Ba_{i+1,j}]_{0\leq i,j\leq n-1} = \sum_{s=0}^{n} A^s B^{n-s} \det[a_{i+\chi(i\geq s),j}]_{0\leq i,j\leq n-1}.$$

where A and B are arbitrary constants.

By combining Theorem 3.1 and Lemma 3.5 we obtain the following generalization of Corollary 5 from [10].

Corollary 3.6. *Let n be a positive integer and $\alpha_0, \alpha_1, \ldots, \alpha_n$ non-negative integers. Then*

$$\det[AC_{\alpha_i+j} + BC_{\alpha_i+j+1}]_{0\leq i,j\leq n-1} = \prod_{0\leq i<j\leq n} (\alpha_j - \alpha_i) \prod_{i=0}^{n-1} \frac{(i+n)!}{(2i)!} \prod_{i=0}^{n} \frac{(2\alpha_i)!}{\alpha_i!(\alpha_i+n)!}$$
$$\times \sum_{s=0}^{n} \frac{A^s B^{n-s} \alpha_s!(\alpha_s+n)!}{(2\alpha_s)! \prod_{j=0}^{s-1}(\alpha_s - \alpha_j) \prod_{j=s+1}^{n}(\alpha_j - \alpha_s)}$$

The following corollaries are special cases of Corollary 3.6, which will be useful in our further considerations. The second corollary provides a direct generalization of the result proven by Cvetković, Rajković and Ivković [5], concerning the Hankel transform evaluation of the sequence $(C_n + C_{n+1})_{n\in\mathbb{N}_0}$.

Corollary 3.7. *For every $t \in \mathbb{N}_0$ and arbitrary constants A and B, the Hankel transform of the sequence $(AC_{n+t} + BC_{n+t+1})_{n\in\mathbb{N}_0}$ is given by*

$$\det[AC_{i+j+t} + BC_{i+j+t+1}]_{0\leq i,j\leq n-1} = \frac{n!(2n+2t)!}{(t+n)!(t+2n)!} \prod_{p=0}^{t-1} \frac{p!(2n+2p)!}{(2p)!(2n+p)!} \sum_{s=0}^{n} \frac{(s+t)!(n+s+t)!}{s!(n-s)!(2s+2t)!} A^s B^{n-s}.$$

Corollary 3.8. *The Hankel transforms of $(\alpha^2 C_n - \beta C_{n+1})_{n\in\mathbb{N}_0}$ and $(\alpha^2 C_{n+1} - \beta C_{n+2})_{n\in\mathbb{N}_0}$ are given by*

$$\det[\alpha^2 C_{i+j} - \beta C_{i+j+1}]_{0\leq i,j\leq n} = \frac{1}{2^{2n+3}\sqrt{\alpha^2-4\beta}} \left[(\alpha + \sqrt{\alpha^2-4\beta})^{2n+3} - (\alpha - \sqrt{\alpha^2-4\beta})^{2n+3}\right]$$
$$\det[\alpha^2 C_{i+j+1} - \beta C_{i+j+2}]_{0\leq i,j\leq n} = \frac{1}{2^{2n+4}\alpha\sqrt{\alpha^2-4\beta}} \left[(\alpha + \sqrt{\alpha^2-4\beta})^{2n+4} - (\alpha - \sqrt{\alpha^2-4\beta})^{2n+4}\right] \tag{8}$$

Proof. Denote by $\left(\hat{h}_n\right)_{n\in\mathbb{N}_0}$ and $\left(\breve{h}_n\right)_{n\in\mathbb{N}_0}$ the Hankel transforms of $(\alpha^2 C_n - \beta C_{n+1})_{n\in\mathbb{N}_0}$ and $(\alpha^2 C_{n+1} - \beta C_{n+2})_{n\in\mathbb{N}_0}$ respectively. Taking $t = 0, 1$ in Corollary 3.2, we obtain:

$$\begin{aligned} \hat{h}_n &= \sum_{s=0}^{n+1} (-1)^{n-s+1} \alpha^{2s} \beta^{n-s+1} \binom{n+s+1}{n-s+1} \\ \breve{h}_n &= \sum_{s=0}^{n+1} (-1)^{n-s+1} \alpha^{2s} \beta^{n-s+1} \binom{n+s+2}{n-s+1} \end{aligned} \tag{9}$$

By direct verification, we conclude that both sequences satisfy the following difference equation

$$z_n - (\alpha^2 - 2\beta) z_{n-1} + \beta^2 z_{n-2} = 0, \qquad (n \geq 2)$$

with initial values $\hat{h}_0 = \alpha^2 - \beta$, $\hat{h}_1 = \alpha^4 - 3\alpha^2\beta + \beta^2$ and $\breve{h}_0 = \alpha^2 - 2\beta$, $\breve{h}_1 = \alpha^4 - 4\alpha^2\beta + 3\beta^2$ respectively. Now expressions (8) follow immediately. □

4 The Hankel transform of aerated shifted Catalan numbers

Consider the sequence of *shifted Catalan numbers* $C^t = (C_n^t)_{n\in\mathbb{N}_0} = (C_{n+t})_{n\in\mathbb{N}_0}$, where $t \in \mathbb{N}_0$ is an arbitrary number. We apply the results of Section 2 to compute the Hankel transform of the aerated sequences $C^{\mathcal{A},t} = \mathcal{A}(C^t)$ and $C^{\mathcal{A},\alpha,t} = \mathcal{A}(C^t; \alpha)$. Note that

$$C^{\mathcal{A},t} = (C_t, 0, C_{t+1}, 0, \ldots), \quad C^{\mathcal{A},\alpha,t} = (\alpha C_t, C_{t+1}, \alpha C_{t+1}, C_{t+2}, \ldots).$$

Direct application of Theorem 2.1, Theorem 2.2, Corollary 3.2 and Corollary 3.7 yields the following Hankel transform evaluations.

Corollary 4.1. *The Hankel transform of the sequence $C^{\mathcal{A},t} = \mathcal{A}(C^t)$ is given by:*

$$\det[C_{i+j}^{\mathcal{A},t}]_{0\leq i,j\leq n} = \begin{cases} \dfrac{(2t)!(2k+1)!}{t!(2k+t+1)!} \displaystyle\prod_{p=0}^{t} \dfrac{p!^2(2k+2p)!^2}{(2p)!^2(2k+p)!^2}, & n = 2k \\ \dfrac{(2t)!(2k+t)!}{t!(2k+2t)!} \displaystyle\prod_{p=0}^{t} \dfrac{p!^2(2k+2p)!^2}{(2p)!^2(2k+p)!^2}, & n = 2k-1 \end{cases}$$

Corollary 4.2. *The Hankel transform of the sequence $C^{\mathcal{A},\alpha,t} = \mathcal{A}(C^t; \alpha)$ is given by*

$$\det[C_{i+j}^{\mathcal{A},\alpha,t}]_{0\leq i,j\leq n} = \begin{cases} \dfrac{2(2k+1)!k!(2t)!(2k+2t+1)!}{t!(k+t)!(2k+t+1)!^2} \displaystyle\prod_{p=0}^{t} \dfrac{p!^2(2k+2p)!^2}{(2p)!^2(2k+p)!^2} \\ \qquad \times \displaystyle\sum_{s=0}^{k} \dfrac{(s+t+1)!(k+s+t+1)!}{s!(k-s)!(2s+2t+2)!} (-1)^{k-s} \alpha^{2s+1}, & n = 2k \\ \dfrac{k!(2t)!}{t!(k+t)!} \displaystyle\prod_{p=0}^{t} \dfrac{p!^2(2k+2p)!^2}{(2p)!^2(2k+p)!^2} \\ \qquad \times \displaystyle\sum_{s=0}^{k} \dfrac{(s+t)!(k+s+t)!}{s!(k-s)!(2s+2t)!} (-1)^{k-s} \alpha^{2s}, & n = 2k-1 \end{cases}$$

Corollary 4.3. *The Hankel-aerating transform of the sequence C^t is given by*

$$\tilde{h}_n = \det[\tilde{\mathbf{H}}_{C^t}]_{(n+1)\times(n+1)} =$$

$$= \begin{cases} \dfrac{t!}{(2t)!}\displaystyle\prod_{p=0}^{t-1}\frac{p!^2(2k+p+t)!}{(2p)!^2(2k+p)!} \\ \qquad \times \displaystyle\sum_{l=0}^{k-1}(-1)^{k+l}\left(\sum_{h=0}^{l}\alpha^{2h}C_{l+t-h}\right)\binom{l+k-1}{2l}\prod_{p=0}^{t}\frac{p+k+l}{2p+2l+1}, & n=2k-1 \\ \dfrac{t!}{(2t)!}\displaystyle\prod_{p=0}^{t-1}\frac{p!^2(2k+p+t+1)!}{(2p)!^2(2k+p+1)!} \\ \qquad \times \displaystyle\sum_{l=1}^{k}(-1)^{k+l-1}\left(\sum_{h=0}^{l-1}\alpha^{2h+1}C_{l+t-h-1}\right)\binom{l+k}{2l}\prod_{p=0}^{t-1}\frac{p+k+l+1}{2p+2l+1}, & n=2k \end{cases}$$

5 Generalization of the series reversion of $\frac{x}{1+\alpha x+\beta x^2}$

In our previous paper [2], we evaluated the Hankel transform of the series reversion of $Q(x) = \frac{x}{1+\alpha x+\beta x^2}$, as well as that of the corresponding shifted sequences. First, let us recall the definition of the series reversion of a (generating) function $f(x)$ which satisfies $f(0)=0$ (see [1]).

Definition 5.1. *For a given (generating) function $v=f(u)$ with the property $f(0)=0$, the series reversion is the sequence $(s_n)_{n\in\mathbb{N}_0}$ such that*

$$u = f^{-1}(v) = s_1 v + s_2 v^2 + \cdots + s_n v^n + \cdots,$$

where $u=f^{-1}(v)$ is the inverse function of $v=f(u)$. Note that since $f(0)=0$, there must hold $s_0 = f^{-1}(0) = 0$.

The general term of the sequence obtained by reverting $Q(x) = \frac{x}{1+\alpha x+\beta x^2}$ is given by (see [1, 2])

$$\sum_{k=0}^{[\frac{n-1}{2}]}\binom{n-1}{2k}C_k\alpha^{n-2k-1}\beta^k. \tag{10}$$

We consider the following generalization $(u_n(t))_{n\in\mathbb{N}_0}$ of that sequence:

$$u_n(t) = \sum_{k=0}^{[\frac{n-1}{2}]}\binom{n-1}{2k}C_{k+t}\alpha^{n-2k-1}\beta^k. \tag{11}$$

Note that (11) reduces to (10) by taking $t=0$. Consider the shifted sequences $(u_n^*(t))_{n\in\mathbb{N}_0}$ and $(u_n^{**}(t))_{n\in\mathbb{N}_0}$ defined by $u_n^*(t) = u_{n+1}(t)$ and $u_n^{**}(t) = u_{n+2}(t)$. Our previous paper [2] provides the evaluation of the corresponding Hankel transforms $h_n^*(0)$, $h_n^{**}(0)$ and $h_n(0)$ using the method based on orthogonal polynomials [5, 12]. The main results in [2] are the following theorems (Theorem 4.3, Theorem 4.4, and Corollary 5.4 in [2]):

Theorem 5.1. [2] *The Hankel transform of the sequence $(u_n^*)_{n\in\mathbb{N}_0}$ is given by*

$$h_n^*(0) = \beta^{\binom{n+1}{2}}. \tag{12}$$

Theorem 5.2. [2] *The Hankel transform of the sequence* $(u_n^{**})_{n\in\mathbb{N}_0}$ *is given by*

$$h_n^{**}(0) = \frac{\beta^{\binom{n+1}{2}}}{2^{n+1}\sqrt{\alpha^2-4\beta}}\left[(\alpha+\sqrt{\alpha^2-4\beta})^{n+2} - (\alpha-\sqrt{\alpha^2-4\beta})^{n+2}\right]. \tag{13}$$

Theorem 5.3. [2] *The Hankel transform of the sequence* $(u_n)_{n\in\mathbb{N}_0}$ *is given by*

$$h_n(0) = \frac{\beta^{\binom{n}{2}}}{2^{n}\sqrt{\alpha^2-4\beta}}\left[\left(\alpha-\sqrt{\alpha^2-4\beta}\right)^{n} - \left(\alpha+\sqrt{\alpha^2-4\beta}\right)^{n}\right]. \tag{14}$$

A different approach to Theorems 5.1-5.2, also based on orthogonal polynomials, is given in a recent paper [4].

In the following sections, we evaluate the Hankel transforms $h_n^*(t)$, $h_n^{**}(t)$ and $h_n(t)$ which provide generalizations of Theorems 5.1-5.3. The proof is based on the application of the falling α-binomial transform (Section 6) and results from the previous sections.

6 The falling α-binomial transform

The following transform is a generalization of the well-known binomial transform and was introduced by Spivey and Steil [14]. We will use it in further considerations.

Definition 6.1. *For a given sequence* $a = (a_n)_{n\in\mathbb{N}_0}$*, its falling* α*-binomial transformation* $b = \mathcal{B}(a;\alpha)$ *is defined to be the sequence*

$$b_n = \sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}a_k.$$

The following lemma provides an extension to the classical result that the Hankel transform is invariant under the binomial transform.

Lemma 6.1. *For an arbitrary sequence* $a = (a_n)_{n\in\mathbb{N}_0}$ *and number* α*, we have* $\mathcal{H}(\mathcal{B}(a;\alpha)) = \mathcal{H}(a)$*.*

The falling α-binomial transform can be written in the following matrix form

$$b = \mathbf{B}^\alpha a, \quad \mathbf{B}^\alpha = \left[\binom{n}{k}\alpha^{n-k}\right]_{n,k\in\mathbb{N}_0}$$

where we treat the sequences a and b as the corresponding column vectors (we also use this notation in the rest of the paper). We call the matrix $\mathbf{B}^\alpha$ the *α-binomial matrix.* The following lemma shows the connection between the Hankel matrices

$$\mathbf{H}_a = [a_{i+j}]_{i,j\in\mathbb{N}_0}, \quad \mathbf{H}_b = [b_{i+j}]_{i,j\in\mathbb{N}_0}$$

and the matrix $\mathbf{B}^\alpha$.

Lemma 6.2. *If* $b = \mathcal{B}(a;\alpha)$ *then we have*

$$\mathbf{H}_b = \mathbf{B}^\alpha\mathbf{H}_a(\mathbf{B}^\alpha)^T. \tag{15}$$

Proof. Let us start from the general element b_{n+m} of the matrix $\mathbf{H}_b$:

$$b_{n+m} = \sum_{t=0}^{n+m} \binom{n+m}{t} \alpha^{n+m-t} a_t.$$

Using the well-known identity $\binom{n+m}{t} = \sum_{k=0}^{n} \binom{n}{k}\binom{m}{t-k}$ we obtain

$$\begin{aligned} b_{n+m} &= \sum_{t=0}^{n+m}\sum_{k=0}^{n} \binom{n}{k}\binom{m}{t-k} \alpha^{n+m-t} a_t = \sum_{l=0}^{m}\sum_{k=0}^{n} \binom{n}{k}\binom{m}{l} \alpha^{n+m-k-l} a_{k+l} \\ &= \sum_{l=0}^{m}\sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} \cdot a_{k+l} \cdot \binom{m}{l} \alpha^{m-l} = \sum_{l=0}^{m}\sum_{k=0}^{n} (\mathbf{B}^{\alpha})_{nk} \cdot a_{k+l} \cdot (\mathbf{B}^{\alpha})_{ml}\,. \end{aligned}$$

This completes the proof of the lemma. □

7 Hankel transform evaluation of sequences $u_n(t)$, $u_n^*(t)$ and $u_n^{**}(t)$

In this section, we show that the sequences $u_n^*(t)$ and $u_n^{**}(t)$ are the falling α-binomial transforms of $\mathcal{A}(c)$ and $\mathcal{A}(c;\alpha)$, where $c = (\beta^n C_{t+n})_{n\in\mathbb{N}_0}$. We also show that $\mathcal{H}(u(t))$ is equal to the Hankel-aerated transform of c. Using this, we can apply the results of Section 2, Section 3 and Section 4 to evaluate the Hankel transforms of $(u_n^*(t))_{n\in\mathbb{N}_0}$, $(u_n^{**}(t))_{n\in\mathbb{N}_0}$ and $(u_n(t))_{n\in\mathbb{N}_0}$. Our main results are Theorems 7.2-7.4. As a special case of these evaluations (for $t = 0$), we re-obtain Theorems 5.1- 5.3, proved in [2].

7.1 The sequence $u_n^*(t)$

Let the sequence $(c_n)_{n\in\mathbb{N}_0}$ be defined by $c_n = \beta^n C_{n+t}$ and let $p = \mathcal{A}(c)$, i.e.

$$p_n = \begin{cases} \beta^k C_{k+t}, & n = 2k \\ 0, & n = 2k-1 \end{cases}.$$

Recall that $u_n^*(t)$ can be expressed as follows (directly from (11)):

$$u_n^*(t) = u_{n+1}(t) = \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} \alpha^{n-2k}\beta^k C_{k+t} = \sum_{l=0}^{n} \binom{n}{l} \alpha^{n-l} p_l$$

which implies that $(u_n^*(t))_{n\in\mathbb{N}_0} = \mathcal{B}(p;\alpha)$ and hence $h^*(t) = \mathcal{H}(p)$ (Lemma 6.1). We also need the following proposition.

Proposition 7.1. *Let $s = (s_n)_{n\in\mathbb{N}_0}$ be an arbitrary sequence and let $h = \mathcal{H}(s)$ be its Hankel transform. Then $\mathcal{H}\left((r^n s_n)_{n\in\mathbb{N}_0}\right) = \left(r^{n(n+1)} h_n\right)_{n\in\mathbb{N}_0}$ where r is an arbitrary number.*

Note that the sequence $p = \mathcal{A}(c)$ can be expressed as $p = \left(\beta^{n/2} C_n^{\mathcal{A},t}\right)_{n\in\mathbb{N}_0}$ where $C^{\mathcal{A},t} = \mathcal{A}(C^t)$ (see Section 4). Hence

$$h_n^*(t) = \det[p_{i+j}]_{0\le i,j\le n} = \beta^{\binom{n+1}{2}} \det[C_{i+j}^{\mathcal{A},t}]_{0\le i,j\le n}. \tag{16}$$

The following theorem is obtained directly from (16) and Corollary 4.1.

Theorem 7.2. *The Hankel transform $(h_n^*(t))_{n\in\mathbb{N}_0}$ of the sequence $(u_n^*(t))_{n\in\mathbb{N}_0}$ is given by:*

$$
\begin{aligned}
h_{2k}^*(t) &= \beta^{\binom{2k+1}{2}} \frac{(2t)!(2k+1)!}{t!(2k+t+1)!} \prod_{p=0}^{t} \frac{p!^2(2k+2p)!^2}{(2p)!^2(2k+p)!^2} \\
h_{2k-1}^*(t) &= \beta^{\binom{2k}{2}} \frac{(2t)!(2k+t)!}{t!(2k+2t)!} \prod_{p=0}^{t} \frac{p!^2(2k+2p)!^2}{(2p)!^2(2k+p)!^2}
\end{aligned}
\tag{17}
$$

Note that by taking $t = 0$ we re-obtain the result $h_n^*(0) = \beta^{\binom{n+1}{2}}$ of Theorem 5.1.

7.2 The sequence $u_n^{**}(t)$

Let $a = \mathcal{A}(c;\alpha)$, i.e. the α-aerating transform of the sequence $c_n = \beta^n C_{n+t}$. The sequence $a = (a_n)_{n\in\mathbb{N}_0}$ can be expressed as follows

$$
a_n = \begin{cases} \alpha\beta^k C_{k+t}, & n = 2k \\ \beta^k C_{k+t}, & n = 2k-1 \end{cases}. \tag{18}
$$

According to (11) we have

$$
u_n^{**}(t) = \sum_{k=0}^{[\frac{n+1}{2}]} \binom{n}{2k-1} \alpha^{n-(2k-1)} \beta^k C_{k+t} + \sum_{k=0}^{[\frac{n+1}{2}]} \binom{n}{2k} \alpha^{n-2k} (\alpha\beta^k C_{k+t}) = \sum_{l=0}^{n} \binom{n}{l} \alpha^{n-l} a_l.
$$

Hence $u^{**}(t) = \mathcal{B}(a;\alpha)$ and from Lemma 6.1 we conclude that $\mathcal{H}(u^{**}(t)) = \mathcal{H}(a)$.

Assume that $n = 2k-1$. According to Theorem 2.2 and Proposition 7.1, we obtain

$$
\begin{aligned}
\det[a_{i+j}]_{0\le i,j\le n} &= \det[\alpha^2\beta^{i+j}C_{i+j+t} - \beta^{i+j+1}C_{i+j+t+1}]_{0\le i,j\le k-1} \cdot \det[\beta^{i+j+1}C_{i+j+t+1}]_{0\le i,j\le k-1} \\
&= \beta^{k(2k-1)} \det[\alpha^2 C_{i+j+t} - \beta C_{i+j+t+1}]_{0\le i,j\le k-1} \cdot \det[C_{i+j+t+1}]_{0\le i,j\le k-1}
\end{aligned}
\tag{19}
$$

Similarly, for $n = 2k$ we have

$$
\begin{aligned}
\det[a_{i+j}]_{0\le i,j\le n} &= \alpha \det[\alpha^2\beta^{i+j+t+1}C_{i+j+t+1} - \beta^{i+j+2}C_{i+j+t+2}]_{0\le i,j\le k-1} \cdot \det[\beta^{i+j+1}C_{i+j+t}]_{0\le i,j\le k} \\
&= \alpha\beta^{k(2k+1)} \det[\alpha^2 C_{i+j+t+1} - \beta C_{i+j+t+2}]_{0\le i,j\le k-1} \cdot \det[C_{i+j+t}]_{0\le i,j\le k}.
\end{aligned}
\tag{20}
$$

Now using Corollary 3.2 and Corollary 3.7 we get the following theorem.

Theorem 7.3. *The Hankel transform $(h_n^{**}(t))_{n\in\mathbb{N}_0}$ of the sequence $(u_n^{**}(t))_{n\in\mathbb{N}_0}$ is given by*

$$
\begin{aligned}
h_{2k-1}^{**}(t) &= \beta^{k(2k-1)} \frac{k!(2t)!}{t!(k+t)!} \prod_{p=0}^{t} \frac{p!^2(2k+2p)!^2}{(2p)!^2(2k+p)!^2} \\
&\quad \times \sum_{s=0}^{k} \frac{(s+t)!(k+s+t)!}{s!(k-s)!(2s+2t)!} (-1)^{k-s}\beta^{k-s}\alpha^{2s}, \\
h_{2k}^{**}(t) &= \beta^{k(2k+1)} \frac{2(2k+1)!k!(2t)!(2k+2t+1)!}{t!(k+t)!(2k+t+1)!^2} \prod_{p=0}^{t} \frac{p!^2(2k+2p)!^2}{(2p)!^2(2k+p)!^2} \\
&\quad \times \sum_{s=0}^{k} \frac{(s+t+1)!(k+s+t+1)!}{s!(k-s)!(2s+2t+2)!} (-1)^{k-s}\beta^{k-s}\alpha^{2s+1}.
\end{aligned}
\tag{21}
$$

Taking $t=0$ in (19) and (20) yields

$$h_n(0) = \begin{cases} \beta^{k(2k-1)} \det[\alpha^2 C_{i+j} - \beta C_{i+j+1}]_{0\le i,j\le k-1}, & n = 2k-1 \\ \alpha\beta^{k(2k+1)} \det[\alpha^2 C_{i+j+1} - \beta C_{i+j+2}]_{0\le i,j\le k-1}, & n = 2k \end{cases}$$

and using Corollary 3.8 we re-obtain Theorem 5.2.

7.3 The sequence $u_n(t)$

We have already proved that $u^*(t) = \mathcal{B}(p;\alpha)$ and $u^{**}(t) = \mathcal{B}(a;\alpha)$. The first equation can be written in matrix notation as $u^*(t) = \mathbf{B}^\alpha p$. Furthermore, Lemma 6.2 yields $\mathbf{H}_{u^{**}(t)} = \mathbf{B}^\alpha \mathbf{H}_a (\mathbf{B}^\alpha)^T$ and the following matrix equality holds:

$$\begin{bmatrix} 1 & \\ & \mathbf{B}^\alpha \end{bmatrix} \begin{bmatrix} 0 & p^T \\ p & \mathbf{H}_a \end{bmatrix} \begin{bmatrix} 1 & \\ & (\mathbf{B}^\alpha)^T \end{bmatrix} = \begin{bmatrix} 0 & p^T(\mathbf{B}^\alpha)^T \\ \mathbf{B}^\alpha p & \mathbf{B}^\alpha \mathbf{H}_a (\mathbf{B}^\alpha)^T \end{bmatrix} = \begin{bmatrix} 0 & (u^*(t))^T \\ u^*(t) & \mathbf{H}_{u^{**}(t)} \end{bmatrix} = \mathbf{H}_{u(t)} \tag{22}$$

Hence, the determinant of the $(n+1)\times(n+1)$ principal minor of $\mathbf{H}_{u(t)}$, formed by the rows and columns with indices $1, 2, \dots, n+1$, is equal to the same minor of the matrix

$$\tilde{\mathbf{H}}_c = \begin{bmatrix} 0 & p^T \\ p & \mathbf{H}_a \end{bmatrix}.$$

In other words, $h_n(t)$ is equal to the n-th member of the Hankel-aerating transform of the sequence $(c_n)_{n\in\mathbb{N}_0}$, and can be evaluated using Theorem 2.3.

Theorem 7.4. *The Hankel transform $(h_n(t))_{n\in\mathbb{N}_0}$ of the sequence $(u_n(t))_{n\in\mathbb{N}_0}$ is given by:*

$$\begin{aligned} h_{2k-1}(t) &= \beta^{(k-1)(2k-1)} \frac{t!}{(2t)!} \prod_{p=0}^{t-1} \frac{p!^2(2k+p+t)!}{(2p)!^2(2k+p)!} \\ &\quad \times \sum_{l=0}^{k-1} (-1)^{k+l} \left(\sum_{h=0}^{l} \alpha^{2h} \beta^{k-1-h} C_{l+t-h} \right) \binom{l+k-1}{2l} \prod_{p=0}^{t} \frac{p+k+l}{2p+2l+1} \\ h_{2k}(t) &= \beta^{k(2k-1)} \frac{t!}{(2t)!} \prod_{p=0}^{t-1} \frac{p!^2(2k+p+t+1)!}{(2p)!^2(2k+p+1)!} \\ &\quad \times \sum_{l=1}^{k} (-1)^{k+l+1} \left(\sum_{h=0}^{l-1} \alpha^{2h+1} \beta^{k-1-h} C_{l+t-h-1} \right) \binom{l+k}{2l} \prod_{p=0}^{t-1} \frac{p+k+l+1}{2p+2l+1} \end{aligned} \tag{23}$$

Proof. We distinguish two cases depending on the parity of n.

Case 1. $n = 2k-1$ is odd. According to Theorem 2.3, it holds that $h_{2k-1}(t) = P_1 \cdot P_2$ where

$$P_1 = \det[c_{i+j}]_{0\le i,j\le k-1} = \det[\beta^{i+j} C_{i+j+t}]_{0\le i,j\le k-1} = \beta^{k(k-1)} \det[C_{i+j+t}]_{0\le i,j\le k-1} \tag{24}$$

and

$$\begin{aligned} P_2 &= \sum_{l=0}^{k-1} (-1)^{k+l} \left(\sum_{h=0}^{l} \alpha^{2h} c_{l-h} \right) \det[c_{i+j+\chi(j\ge l)+1}]_{0\le i,j\le k-2} \\ &= \sum_{l=0}^{k-1} (-1)^{k+l} \left(\sum_{h=0}^{l} \alpha^{2h} \beta^{l-h} C_{l+t-h} \right) \beta^{k^2-k-l} \det[C_{i+j+\chi(j\ge l)+t+1}]_{0\le i,j\le k-2}. \end{aligned} \tag{25}$$

Using Corollary 3.2 and Corollary 3.4, together with (24) and (25), we get the expression for $h_{2k-1}(t)$ in (23).

Case 2. $n = 2k$ is even. Now $h_{2k}(t) = P_1 \cdot P_2$ where

$$P_1 = \det[\beta^{i+j+1} C_{i+j+t+1}]_{0\le i,j\le k-1} = \beta^{k^2} \det[C_{i+j+t+1}]_{0\le i,j\le k-1} \tag{26}$$

and

$$\begin{aligned} P_2 &= \sum_{l=1}^{k} (-1)^{k+l+1} \left(\sum_{h=0}^{l-1} \alpha^{2h+1} c_{l-1-h} \right) \det[c_{i+j+\chi(j\ge l)}]_{0\le i,j\le k-1} \\ &= \sum_{l=1}^{k} (-1)^{k+l+1} \left(\sum_{h=0}^{l-1} \alpha^{2h+1} \beta^{l-1-h} C_{l+t-1-h} \right) \beta^{k^2-l} \det[C_{i+j+\chi(j\ge l)+t}]_{0\le i,j\le k-1}. \end{aligned} \tag{27}$$

Similarly, using Corollary 3.2 and Corollary 3.3, together with (26) and (27), we get the expression for $h_{2k}(t)$ in (23). $\square$

In the special case $t = 0$, expression (23) reduces to:

$$\begin{aligned} h_{2k-1}(0) &= \beta^{(k-1)(2k-1)} \sum_{l=0}^{k-1} (-1)^{k+l} \left(\sum_{h=0}^{l} \alpha^{2h} \beta^{k-h-1} C_{l-h} \right) \binom{l+k}{2l+1} \\ h_{2k}(0) &= \beta^{k(2k-1)} \sum_{l=1}^{k} (-1)^{k+l+1} \left(\sum_{h=0}^{l-1} \alpha^{2h+1} \beta^{k-h-1} C_{l-1-h} \right) \binom{l+k}{2l} \end{aligned} \tag{28}$$

Proof of Theorem 5.3. We can rewrite expressions (28) as follows (exchanging the order of summation):

$$\begin{aligned} h_{2k-1}(0) &= \beta^{(k-1)(2k-1)} \sum_{h=0}^{k-1} \alpha^{2h} \beta^{k-1-h} \sum_{l=h}^{k-1} (-1)^{k+l} C_{l-h} \binom{l+k}{2l+1} \\ h_{2k}(0) &= \beta^{k(2k-1)} \sum_{h=0}^{k-1} \alpha^{2h+1} \beta^{k-1-h} \sum_{l=h+1}^{k} (-1)^{k+l+1} C_{l-1-h} \binom{l+k}{2l} \end{aligned} \tag{29}$$

Now let $z_n = \beta^{-\binom{n}{2}} h_n(0)$ and in the second equation of (29) decrease the bounds for l by 1. By direct verification we conclude that z_n satisfies the three-term linear difference equation $z_{n+2} - \alpha z_{n+1} + \beta z_n = 0$ for all $n \in \mathbb{N}_0$, which directly implies the expression (14) in Theorem 5.3:

$$h_n(0) = \beta^{\binom{n}{2}} z_n = \frac{\beta^{\binom{n}{2}}}{2^n \sqrt{\alpha^2 - 4\beta}} \left[\left(\alpha - \sqrt{\alpha^2 - 4\beta} \right)^n - \left(\alpha + \sqrt{\alpha^2 - 4\beta} \right)^n \right].$$

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