Re-nnd solutions of the matrix equation $AXB = C$

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Abstract

In this article we consider Re-nnd solutions of the equation $AXB = C$ with respect to $X$, where $A, B, C$ are given matrices. We give necessary and sufficient conditions for the existence of Re-nnd solutions and present a general form of such solutions. As a special case when $A = I$ we obtain the results from paper of J. Gross (Explicit solutions to the matrix inverse problem $AX = B$, Linear Algebra Appl. 289 131-134 (1999)).

1 Introduction

Let $C^{n \times m}$ denote the set of complex $n \times m$ matrices. $I_n$ denotes the unit matrix of order $n$. By $A^*$, $\mathcal{R}(A)$, $\text{rank}(A)$ and $\mathcal{N}(A)$, we denote the conjugate transpose, the range, the rank and the null space of $A \in C^{n \times m}$.

The Hermitian part of $X$ is defined as $H(X) = \frac{1}{2}(X + X^*)$. We will say that $X$ is Re-nnd (Re-nonnegative definite) if $H(X) \geq 0$ and $X$ is Re-pd (Re-positive definite) if $H(X) > 0$.

The symbol $A^-$ stands for an arbitrary generalized inner inverse of $A$, i.e. $A^-$ satisfies $AA^- A = A$. By $A^\dagger$ we denote the Moore-Penrose inverse of $A \in C^{n \times m}$, i.e. the unique matrix $A^\dagger \in C^{m \times n}$ satisfying

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$
For some important properties of generalized inverses see [5] and [6].

Many authors have studied the well-known equation

\[ AXB = C \]  

with the unknown matrix \( X \), such that \( X \) belongs to some special class of matrices. For example, in [19] and [7] the existence of reflexive and anti-reflexive, with respect to a generalized reflection matrix \( P \), solutions of the matrix equation (1) was considered, while in [15], [9], [18], [20] necessary and sufficient conditions for the existence of symmetric and antisymmetric solutions of the equation (1) were investigated.

The Hermitian nonnegative definite solutions for the equation \( AXA^* = B \) were investigated by C. G. Khatri and S. K. Mitra [15], J.K. Baksalary [4], H. Dai and P. Lancaster [10], J. Gross [12], X. Zhang and M. Y. Cheng [24], X. Zhang [25].

L.Wu [22] studied Re-positive definite solutions of the equation \( AX = C \) and L. Wu and B. Cain [23] found the set of all complex Re-nnd matrices \( X \) such that \( XB = C \) and presented a criterion for Re-nndness. J. Gross [11] gave an alternative approach, which simultaneously delivers explicit Re-nnd solutions and gave a corrected version of some results from [23]. Some results from [23] were extended in the paper of Q. Wang and C. Yang [21], in which authors presented criteria for \( 2 \times 2 \) and \( 3 \times 3 \) partitioned matrices to be Re-nnd, found necessary and sufficient conditions for the existence of Re-nnd solutions of the equation (1) and derived an expression for these solutions. In the paper of A. Dajić and J. Koliha [3], for the first time, a general form of Re-nnd solutions of the equation \( AX = B \) is given, where \( A \) and \( B \) are given operators between Hilbert spaces. Beside these papers many other papers have dealt with the problem of finding the Re-nnd and Re-pd solutions of some other forms of equations.

In this paper, we first consider the matrix equation

\[ AXA^* = C, \]

where \( A \in C^{n \times m}, C \in C^{n \times n} \), and find necessary and sufficient conditions for the existence of Re-nnd solutions. Also, we present a general form of these solutions. Using this result, we get necessary and sufficient conditions for the equation

\[ AXB = C, \]

where \( A \in C^{n \times m}, B \in C^{m \times n} \) and \( C \in C^{n \times n} \), to have a Re-nnd solution. This way, the results of [23] and [11] follow as a corollary and a general
form of those solutions is given in addition. As far as the author is aware, this is the first time necessary and sufficient conditions for the existence of a Re-nnd solutions of the equation $AXB = C$ has been given in terms of g-inverses.

Now, we will state some well-known results which will be very often used in the next section.

**Theorem 1.1 ([17])** Let $A \in C^{n \times m}$, $B \in C^{p \times r}$ and $C \in C^{n \times r}$. Then the matrix equation

$$AXB = C$$

is consistent if and only if, for some $A^{-}, B^{-}$,

$$AA^{-}CB^{-}B = C,$$  \hspace{1cm} (2)

in which case the general solution is

$$X = A^{-}CB^{-} + Y - A^{-}AYBB^{-},$$  \hspace{1cm} (3)

for arbitrary $Y \in C^{m \times p}$.

The following result was derived by Albert [1] for block matrices, by Cvetković-Ilić et al [8] for $C^{*}$ algebras, for the special case of the Moore-Penrose inverse and by Dajić and Koliha [3] for operators between different Hilbert spaces. Here, we will give the basic version proved in [1].

**Theorem 1.2** Let $M \in C^{(n+m) \times (n+m)}$ be a hermitian block-matrix given by

$$M = \begin{bmatrix} A & B \\ B^{*} & D \end{bmatrix},$$

where $A \in C^{n \times n}$ and $D \in C^{m \times m}$. Then, $M \geq 0$ if and only if

$$A \geq 0, \quad AA^{\dagger}B = B, \quad D - B^{*}A^{\dagger}B \geq 0.$$  

Anderson and Duffin [2] define parallel sum of matrices, for a pair of matrices of the same order as

$$A : B = A(A + B)^{-}B.$$  

It is clear that for this definition to be meaningful, the expression $A(A + B)^{-}B$ must be independent of the choice of the g-inverse $(A+B)^{-}$. Hence, a
pair of matrices $A$ and $B$ will be said to be parallel summable if $A(A+B)^{-}B$ is invariant under the choice of the inverse $(A+B)^{-}$, that is, if

$$
\mathcal{R}(A) \subseteq \mathcal{R}(A + B), \quad \mathcal{R}(A^*) \subseteq \mathcal{R}(A^* + B^*),
$$
or equivalently

$$
\mathcal{R}(B) \subseteq \mathcal{R}(A + B), \quad \mathcal{R}(B^*) \subseteq \mathcal{R}(A^* + B^*).
$$
Note that

$$
\mathcal{R}(A) \subseteq \mathcal{R}(B) \iff BB^-A = A.
$$

By Theorem 2.1 [13],

$$
\mathcal{R}(A) \subseteq \mathcal{R}(B) \iff AA^* \leq \lambda^2 BB^*, \text{ for some } \lambda \geq 0,
$$
so for the non-negative definite matrices $A$ and $B$, we have that

$$
A \leq A + B \iff \mathcal{R}(A^{1/2}) \subseteq \mathcal{R}((A + B)^{1/2}),
$$
which implies $\mathcal{R}(A) \subseteq \mathcal{R}((A + B)^{1/2})$ or equivalently

$$
(A + B)^{1/2}((A + B)^{1/2})^\dagger A = A.
$$

Now,

$$
(A + B)(A + B)^\dagger A = ((A + B)^{1/2}((A + B)^{1/2})^\dagger)^2 A = A,
$$
which is equivalent to $\mathcal{R}(A) \subseteq \mathcal{R}(A + B)$.

Hence, non-negative definite matrices $A$ and $B$ are parallel summable. Furthermore, in [2] it was proved that for a pair of parallel summable matrices holds

$$
A : B = B : A,
$$
i.e.

$$
A(A + B)^{-}B = B(A + B)^{-}A. \tag{4}
$$
2 Results

Next result was first proved by L. Wu and B. Cain [23] and later restated by J. Gross [11]. It gives necessary and sufficient conditions for the matrix equation $AX = B$ to have a Re-nnd solution $X$, where $A, B$ are given matrices of suitable size and presents a possible explicit expression for $X$.

**Theorem 2.1** Let $A \in C^{n \times m}$, $B \in C^{n \times m}$. There exists a Re-nnd matrix $X \in C^{m \times m}$ satisfying $AX = B$ if and only if $AA^\dagger B = B$ and $AB^*$ is Re-nnd.

From the proof of this theorem we can see that

$$X_0 = A^\dagger C - (A^\dagger C)^* + A^\dagger AC^* (A^\dagger)^*$$

is one of Re-nnd solutions of $AX = B$. Also, in the [11] the author mentions that any matrix of the form

$$X = X_0 + (I - A^\dagger A)Y(I - A^\dagger A),$$

with $Y \in C^{m \times m}$ which is Re-nnd is also Re-nnd solution of $AX = B$, in the case where such solutions exist, but he didn’t present a general form of such solutions. Our main aim is to generalize these results to the equation $AXB = C$ and to present a general form of Re-nnd solutions of it.

First, we will consider the equation

$$AXA^* = C$$

and find necessary and sufficient conditions for the existence of Re-nnd solutions.

The next auxiliary result presents a general form of a solution $X$ of (5) which satisfies $H(X) = 0$.

**Lemma 2.1** If $A \in C^{n \times m}$, then $X \in C^{m \times m}$ is a solution of the equation

$$AXA^* = 0$$

which satisfies $H(X) = 0$ if and only if

$$X = W(I - A^\dagger A) - (I - A^\dagger A)W^*,$$

for some $W \in C^{m \times m}$. 

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Proof. Denote by \( r = \text{rank}(A) \). Let us suppose that \( X \) is a solution of the equation (6) and \( H(X) = 0 \). Using a singular value decomposition of \( A = U^* \text{Diag}(D,0)V \), where \( U \in \mathbb{C}^{n \times n} \), \( V \in \mathbb{C}^{m \times m} \) are unitary and \( D \in \mathbb{C}^{r \times r} \) is an invertible matrix, we have that

\[
A^\dagger = V^* \text{Diag}(D^{-1},0)U \quad \text{and} \quad X = V^* \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} V,
\]

for some \( X_1 \in \mathbb{C}^{r \times r} \) and \( X_4 \in \mathbb{C}^{(m-r) \times (m-r)} \).

From \( AXA^* = 0 \) we get that \( X_1 = 0 \) and, by \( H(X) = 0 \), that \( X_3 = -X_2^* \) and \( H(X_4) = 0 \). Hence,

\[
X = V^* \begin{bmatrix} 0 & X_2 \\ -X_2^* & X_4 \end{bmatrix} V.
\]

Taking into account that \( H(X_4) = 0 \), for \( W = V^* \begin{bmatrix} I & X_2 \\ 0 & X_4/2 \end{bmatrix} V \), we have

\[
X = W(I - A^\dagger A) - (I - A^\dagger A)W^*.
\]

In the other direction we can easily check that for arbitrary \( W \in \mathbb{C}^{m \times m} \), \( X \) defined by (7) is a solution of the equation (6) which satisfies \( H(X) = 0 \). \( \square \)

**Theorem 2.2** Let \( A \in \mathbb{C}^{m \times m} \), \( C \in \mathbb{C}^{n \times n} \) be given matrices such that the equation (5) is consistent and let \( r = \text{rank}H(C) \). There exists a Re-nnd solution of the equation (5) if and only if \( C \) is Re-nnd. In this case the general Re-nnd solution is given by

\[
X = A^n C(A^n)^* + (I - A^-A)UU^*(I - A^-A)^* + W(I - A^\dagger A) - (I - A^\dagger A)W^*
\]

with

\[
A^n = A^- + (I - A^-A)Z((H(C))^{1/2})^-,
\]

where \( A^-,(H(C))^{1/2}^- \) are arbitrary but fixed generalized inverses of \( A \) and \( (H(C))^{1/2} \), respectively, \( Z \in \mathbb{C}^{m \times n} \), \( U \in \mathbb{C}^{m \times (m-r)} \), \( W \in \mathbb{C}^{m \times m} \) are arbitrary matrices.

Proof. If \( X \) is a Re-nnd solution of the equation (5), then

\[
AH(X)A^* = H(C) \geq 0.
\]
In the other direction, if \( C \) is Re-nnd, then \( X_0 = A^−C(A^−)^* \) is Re-nnd solution of the equation (5).

Let us prove that a representation of the general Re-nnd solution is given by (8). If \( X \) is defined by (8), then \( X \) is Re-nnd and \( AXA^* = AA^−C(AA^−)^* = C \).

If \( X \) is an arbitrary Re-nnd solution of (5), then \( H(X) \) is a hermitian nonnegative-definite solution of the equation

\[
AZA^* = H(C),
\]

so, by Theorem 1 [12],

\[
H(X) = A^=H(C)(A^=)^* + (I − A^−A)UU^*(I − A^−A)^*,
\]

where \( A^= \) is given by (9), for some \( Z \in C^{m\times n} \) and \( U \in C^{m\times(m−r)} \).

Note that,

\[
H(X) = H(A^=C(A^=)^* + (I − A^−A)UU^*(I − A^−A)^*),
\]

implying

\[
X = A^=C(A^=)^* + (I − A^−A)UU^*(I − A^−A)^* + Z,
\]

where \( H(Z) = 0 \) and \( AZA^* = 0 \). Using Lemma 2.1, we have that

\[
Z = W(I − A^†A) − (I − A^†A)W^*,
\]

for some \( W \in C^{m\times n} \). Hence, we get that \( X \) has a representation as in (8).

Now, let us consider the equation

\[
AXB = C \tag{10}
\]

where \( A \in C^{n\times m} \), \( B \in C^{m\times n} \) and \( C \in C^{n\times n} \) are given matrices and find necessary and sufficient conditions for the existence of a Re-nnd solution.

Without loss of generality we may assume that \( n = m \) and that matrices \( A \) and \( B \) are both nonnegative definite. This follows from the fact that whenever the equation (10) is solvable then \( X \) is a solution of that equation if and only if \( X \) is a solution of the equation \( A^AXBB^* = A^CB^* \). Hence, from now on, we assume that \( A \) and \( B \) are nonnegative definite matrices from the space \( C^{n\times n} \).

The next theorem is the main result of this paper which presents necessary and sufficient conditions for the equation (10) to have a Re-nnd solution.
Theorem 2.3 Let $A, B, C \in C^{n \times n}$ be given matrices such that equation (10) is solvable. There exists a Re-nnd solution of (10) if and only if

$$T = B(A + B)^{-}C(A + B)^{-}A$$

(11)
is Re-nnd, where $(A + B)^{-}$ is a g-inverse of $A + B$. In this case a general Re-nnd solution is given by

$$X = (A + B)^{*}(C + Y + Z + W)((A + B)^{*})^{*}$$
$$+ (I - (A + B)^{-}(A + B))UU^{*}(I - (A + B)^{-}(A + B))^{*}$$
$$+ Q(I - (A + B)^{†}(A + B)) - (I - (A + B)^{†}(A + B))Q^{*},$$

(12)

where $Y, Z, W$ are arbitrary solutions of the equations

$$Y(A + B)^{-}B = C(A + B)^{-}A,$$
$$A(A + B)^{-}Z = B(A + B)^{-}C,$$
$$A(A + B)^{-}W(A + B)^{-}B = T,$$

(13)
such that $C + Y + Z + W$ is Re-nnd, $(A + B)^{*}$ is defined by

$$(A + B)^{*} = (A + B)^{-} + (I - (A + B)^{-}(A + B))P((H(C + Y + Z + W)^{1/2})^{-},$$

where $U \in C^{n \times (n-r)}$, $Q \in C^{n \times n}$, $P \in C^{n \times n}$ are arbitrary, $r = \text{rank}(C + Y + Z + W)$.

Proof. Denote by

$$E = (A + B)^{-}B, \quad F = C(A + B)^{-}A,$$
$$K = A(A + B)^{-}, \quad L = B(A + B)^{-}C.$$

Now, equations (13) are equivalent to

$$YE = F, \quad KZ = L, \quad KWE = T.$$

(14)

Using (4) and the fact that $E$ is g-invertible and $E^{-} = B^{-}(A + B)$, we have that

$$FE^{-}E = C(A + B)^{-}AB^{-}(A + B)(A + B)^{-}B$$
$$= C(A + B)^{-}AB^{-}B = CB^{-}B(A + B)^{-}AB^{-}B$$
$$= CB^{-}A(A + B)^{-}BB^{-}B = CB^{-}A(A + B)^{-}B$$
$$= CB^{-}B(A + B)^{-}A = C(A + B)^{-}A = F,$$
which implies that the equation \( YE = F \) is consistent. In a similar way, we can prove that the other two equations from (14) are consistent. Furthermore, \( T^* = F^*E = KL^* \) is Re-nnd which implies, by Theorem 2.1, that the first two equations from (14) have Re-nnd solutions.

Now, suppose that the equation (10) has a Re-nnd solution \( X \). Then

\[
H(T) = H(B(A + B)AXB(A + B)A) \\
= (B(A + B)A)H(X)(B(A + B)A)^* \geq 0.
\]

Conversely, let \( T \) be Re-nnd. We can check that

\[
X_0 = (A + B)(C + Y + Z + W)(A + B)^\text{−}
\]

is a solution of the equation (10), where \( Y, Z, W \) are arbitrary solutions of the equations (14). This follows from

\[
AX_0B = (A + B)(A + B)^\text{−}C(A + B)^\text{−}(A + B) \\
= (A + B)(A + B)^\text{−}AA^\text{−}CB^\text{−}B(A + B)^\text{−}(A + B) \\
= AA^\text{−}CB^\text{−}B = C.
\]

Now, we have to prove that for some choice of \( Y, Z, W \), matrix \( C + Y + Z + W \) is Re-nnd which would imply that \( X_0 \) is Re-nnd.

Let

\[
Y = FE^\text{−} - (FE^\text{−})^* + (E^\text{−})^*F^*EE^\text{−} + (I - EE^\text{−})^*(I - EE^\text{−})^*, \\
Z = K^\text{−}L - (K^\text{−}L)^* + K^\text{−}KL^*(K^\text{−})^* + (I - K^\text{−}K)Q(I - K^\text{−}K)^*, \\
W = K^\text{−}TE^\text{−} - (I - K^\text{−}K)S - S(I - EE^\text{−}),
\]

where \( Q = (C^* - K^\text{−}T^*E^\text{−})(C^* - K^\text{−}T^*E^\text{−})^* \) and \( S = K^\text{−}KC^* + C^*EE^\text{−} \). Obviously, \( Y, Z, W \) are solutions of the equations (14) and

\[
H(Y) = (E^\text{−})^*H(T)E^\text{−} + (I - EE^\text{−})^*(I - EE^\text{−}), \\
H(Z) = K^\text{−}H(T)(K^\text{−})^* + (I - K^\text{−}K)H(Q)(I - K^\text{−}K)^* \\
H(W) = K^\text{−}TE^\text{−} + (E^\text{−})^*T^*(K^\text{−})^* \\
- H(C^*EE^\text{−} + K^\text{−}KC^* - 2K^\text{−}T^*E^\text{−}).
\]

Using

\[
K^\text{−}KK^\text{−}T^*E^\text{−} = K^\text{−}KK^\text{−}KL^*E^\text{−} = K^\text{−}KL^*E^\text{−} = K^\text{−}T^*E^\text{−}, \\
K^\text{−}T^*E^\text{−}EE^\text{−} = K^\text{−}F^*EE^\text{−}EE^\text{−} = K^\text{−}F^*EE^\text{−} = K^\text{−}T^*E^\text{−}, \\
KC^*E = KL^* = T^*.
\]
we compute,

\[ H(C + Y + Z + W) = ((E^-)^* + K^-)H(T)((E^-)^* + K^-)^* \]

\[ + \begin{bmatrix} (I - EE^-)^* & (I - K^-K) \end{bmatrix} D \begin{bmatrix} I - EE^- \\ (I - K^-K)^* \end{bmatrix}, \]

where \( D = \begin{bmatrix} I & C - (E^-)^*T(K^-)^* \\ C^* - K^-T^*E^- & H(Q) \end{bmatrix}. \) By Theorem 1.2 it follows that \( D \) is nonnegative definite, so \( H(C + Y + Z + W) \geq 0. \)

Hence, with such a choice of \( Y, Z, W \), it can be seen that \( X_0 \) defined by (15) is Re-nnd solution of (10). So, we proved the sufficient part of the theorem.

Let \( X \) be an arbitrary Re-nnd solution of (10). It is evident that \( Y = AXA, Z = BXB \) and \( W = BXA \) are solutions of (14), and that

\[(A + B)X(A + B) = C + Y + Z + W\]

is Re-nnd. Now, using Theorem 2.2 we get that \( X \) has the representation (12). □

Let us mention that, if we apply Theorem 2.3 to the equation

\[ AX = C, \]

we get as a corollary, the Theorem 2.1 from [11].

Note that if the equation \( AX = C \) is consistent then \( X \) is a solution of it if and only if \( A^*AX = A^*C \). By Theorem 2.3, we get that there exists a Re-nnd solution of the equation \( AX = C \) if and only if

\[ T = (A^*A + I)^{-1}A^*C(A^*A + I)^{-1}A^*A \]

is Re-nnd. Note that in this case \((I + A^*A)\) is invertible matrix.

Let us prove that \( T \) is Re-nnd if and only if \( CA^* \) is Re-nnd. By

\[(A^*A + I)^{-1}A^*A = A^*A(A^*A + I)^{-1}, \]

we have that

\[ T = (A^*A + I)^{-1}A^*(CA^*)((A^*A + I)^{-1}A^*)^*, \]

i.e.

\[ H(T) = ((A^*A + I)^{-1}A^*)H(CA^*)((A^*A + I)^{-1}A^*)^*. \]
From the last equality, $H(CA^*) \geq 0 \Rightarrow H(T) \geq 0$.

Now, suppose that $H(T) \geq 0$. Because of the consistence of the equation $AX = C$, it follows that $AA^\dagger C = C$ which implies that

\[
(A^\dagger)^*(A^* A + I)T((A^\dagger)^*(A^* A + I))^* \\
= (A^\dagger)^* A^* CA^* A A^\dagger = AA^\dagger CA^* = CA^*
\]

i.e.

\[
H(CA^*) = ((A^\dagger)^*(A^* A + I))H(T)((A^\dagger)^*(A^* A + I))^* \geq 0.
\]

References


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