Weighted generalized inverses of partitioned matrices in Banachiewicz-Schur form

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Abstract

In this paper the conditions under which the weighted generalized inverses $A^{(1,4M)}$, $A^{(1,4N)}$, $A^†_{M,N}$ and $A^{d,W}$ can be expressed in Banachiewicz-Schur form are considered and some interesting results are established. This contributes to certain recent results obtained by J.K.Baksalary and G.P.Styan [2] and Y.Wei[15] and it is an extension of their works.

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1 Introduction

Let $C^{n \times m}$ denote the set of all complex $n \times m$ matrices. $I_n$ denotes the unit matrix of order $n$. By $A^* \in C^{m \times n}$ we denote the conjugate transpose matrix of $A \in C^{n \times m}$. Let us recall that the Moore-Penrose inverse of $A \in C^{n \times m}$ is the unique matrix $A^\dagger \in C^{m \times n}$ which satisfies

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.$$  

The Drazin inverse of $A \in C^{n \times n}$ is the matrix $A_D \in C^{n \times n}$ which satisfies

$$A^{k+1}X = A^k, \quad XAX = X, \quad AX =XA,$$

for some nonnegative integer $k$. The least $k$ is the index of $A$, denoted by $\text{ind}(A)$. Generalizing the Moore-Penrose and the Drazin inverse, the weighted Moore-Penrose inverse and the weighted Drazin inverse are defined as follows:

**Definition 1.1** Let $A \in C^{n \times m}$ and let $M \in C^{n \times n}$ and $N \in C^{m \times m}$ be positive definite. The unique matrix $X \in C^{m \times n}$ which satisfies

$$AXA = A, \quad XAX = X, \quad (MAX)^* = MAX, \quad (NXA)^* = NXA, \quad (1)$$

is called the weighted Moore-Penrose inverse of $A$ and it is denoted by $A_{M,N}^\dagger$.

**Definition 1.2** If $A \in C^{n \times m}$ and $W \in C^{m \times n}$ are complex matrices, then the unique solution $X \in C^{n \times m}$ of the equations

$$(AW)^{k+1}XW = (AW)^k, \quad XWAWX = X, \quad AWX = XWA, \quad (2)$$

where $k = \text{ind}(AW)$, is called the $W$-weighted Drazin inverse of $A$ and it is denoted by $A_{d,w}^d$.

Obviously for $M = I_n$ and $N = I_m$ the weighted Moore-Penrose inverse of $A$ is the Moore-Penrose inverse of $A$. If $m = n$ and $W = I_n$, then matrix $X$ which satisfies (2) is the Drazin inverse of $A$. It is well-known that $A_{M,N}^\dagger = N^{-1/2}(M^{1/2}AN^{-1/2})^\dagger M^{1/2}$ and $A_{d,w}^d = [(AW)^D]^2A$. Some interesting properties of weighted Moore-Penrose and the weighted Drazin inverse, among other papers, are investigated in [9], [13].
For $A \in C^{m \times n}$, the set of inner, outer, least-squares weighted generalized and minimum-norm weighted generalized inverses, respectively are given by:

\[
A\{1\} = \{ X \in C^{m \times n} : AXA = A \},
A\{2\} = \{ X \in C^{m \times n} : XAX = X \},
A\{1,3(M)\} = \{ X \in C^{m \times n} : AXA = A, (MAX)^* = MAX \},
A\{1,4(N)\} = \{ X \in C^{m \times n} : AXA = A, (NXA)^* = NXA \},
\]

where $M \in C^{n \times n}$ and $N \in C^{m \times m}$ are positive definite matrices.

In this paper we consider matrix $A \in C^{(m+p) \times (n+q)}$ partitioned as

\[
A = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix},
\]

(3)

where $A_{11} \in C^{m \times n}$ and $A_{22} \in C^{p \times q}$. We use the following definition of the generalized Schur complement.

**Definition 1.3** For a matrix $A \in C^{(m+p) \times (n+q)}$ given by (3) the generalized Schur complement of $A$ in symbol $S(A)$, is defined by

\[
S(A) = A_{22} - A_{21}A_{11}^\alpha A_{12},
\]

(4)

where $A_{11}^\alpha \in A_{11}\{1\}$.

The case $A_{11}^{-1}$ instead of $A_{11}^\alpha$, under assumption that $A_{11}$ is invertible, was first used by Schur [14]. The idea of Schur complements goes back to Sylvester (1851) and the term Schur complements was introduced by E. Haynsworth [10]. Carlson et al. [4] defined the generalized Schur complement by replacing the ordinary inverse with the Moore-Penrose inverse.

The Schur complement and the generalized Schur complement, were studied by a number of authors, including their applications in statistics, matrix theory, electrical network theory, discrete-time regulator problem, sophisticated techniques and some other fields. For interesting results concerning Schur complements see also [1], [5], [6], [7], [8], [12].

Banachiewicz [3] expressed the inverse of a partitioned matrix in terms of Schur complement. When the partitioned matrix $A$, given by (3), is nonsingular and $A_{11}$ is also nonsingular, then $S(A)$ is nonsingular and

\[
A^{-1} = \begin{bmatrix}
A_{11}^{-1} + A_{11}^{-1}A_{12}S^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}S^{-1} \\
-S^{-1}A_{21}A_{11}^{-1} & S^{-1}
\end{bmatrix},
\]

(4)
where we use $S$ instead of $S(A)$.

The motivations for our research are the following:

(1) the paper of Baksalary and Styan [2] in which they extended the result of Marsaglia and Styan [11], considering the necessary and sufficient conditions such that the outer inverses, least-squares generalized inverses and minimum norm generalized inverses can be represented by the Banachiewicz-Schur form;

(2) the paper of Y.Wei [15] in which he found the sufficient conditions for the Drazin inverse to be represented by the Banachiewicz-Schur form.

Our purpose is to generalized these results for the weighted Moore-Penrose inverse and the weighted Drazin inverse of $A$.

2 Results

Let $X \in C^{(n+q)\times(m+p)}$ be given by

$$X = \begin{bmatrix} A^\alpha_{11} + A_{11}^\alpha A_{12} S^\alpha A_{21}^\alpha A_{11}^\alpha & -A_{11}^\alpha A_{12} S^\alpha \\ -S^\alpha A_{21} A_{11}^\alpha & S^\alpha \end{bmatrix},$$

(5)

where $A_{11}^\alpha \in A_{11}\{1\}$ and the positive definite matrices $M \in C^{(m+p)\times(m+p)}$ and $N \in C^{(n+q)\times(n+q)}$ are given by

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix},$$

(6)

where $M_{11} \in C^{m\times m}$, $M_{22} \in C^{p\times p}$, $N_{11} \in C^{n\times n}$, $N_{22} \in C^{q\times q}$.

We begin with the following result of Baksalary and Styan [2], adopting the following notations from [2]:

$$E_{A_{11}} = I - A_{11}^\alpha A_{11}, \quad F_{A_{11}} = I - A_{11} A_{11}^\alpha, \quad E_S = I - S^\alpha S, \quad F_S = I - SS^\alpha,$$

where $S = S(A)$ is the Schur complement of $A$ which is defined in (4) and $S^\alpha \in C^{q\times p}$.

**Theorem 2.1** Let $A$ and $X$ are given by (3) and (5). Then $X \in A\{1\}$ if and only if $S^\alpha \in S\{1\}$ and

$$F_{A_{11}} A_{12} E_S = 0, \quad F_S A_{21} E_{A_{11}} = 0, \quad F_{A_{11}} A_{12} S^\alpha A_{21} E_{A_{11}} = 0.$$

(7)

The last three conditions being independent of the choice of $A_{11}^\alpha \in A_{11}\{1\}$ and $S^\alpha \in S\{1\}$ involved in $E_{A_{11}}, F_{A_{11}}, E_S$ and $F_S$. 


The following theorem give the necessary and sufficient conditions such that \( X \in A\{1, 3(M)\} \) under some conditions.

**Theorem 2.2** If \( M \) is a positive definite matrix given by (6), such that

\[
(M_{12}S\alpha)^* = M_{12}^*A_{11}^0A_{11}^0 - M_{12}^*F_{A_{11}}A_{12}^0A_{21}^0A_{11}^0 + M_{12}^*F_A^*A_{12}^0A_{11}^0, \\
S\alpha M_{22}F_S = E_SM_{22}S\alpha,
\]

then \( X \in A\{1, 3(M)\} \) if and only if \( A_{11}^0 \in A_{11}\{1, 3(M_{11})\} \), \( S\alpha \in S\{1, 3(M_{22})\} \) and

\[
F_{A_{11}}A_{12} = 0, \quad F_SA_{21} = 0.
\]

The last two conditions are independent of the choice of \( A_{11}^0 \in A_{11}\{1\} \) and \( S\alpha \in S\{1\} \) involved in \( F_{A_{11}} \) and \( F_S \).

**Proof.** Suppose that \( A_{11}^0 \in A_{11}\{1, 3(M_{11})\} \), \( S\alpha \in S\{1, 3(M_{22})\} \) and the conditions (9) are satisfied. Then the conditions from the Theorem 2.1 are satisfied and \( X \) is an inner inverse of \( A \). Also, we have that

\[
(MAX)_{11} = M_{11}A_{11}^0A_{11}^0 - M_{11}F_{A_{11}}A_{12}^0A_{21}^0A_{11}^0 + M_{12}^*F_SA_{21}^0A_{11}^0, \\
(MAX)_{12} = M_{11}F_{A_{11}}A_{12}^0A_{21}^0A_{11}^0 + M_{12}^*F_A^*A_{12}^0A_{11}^0 = M_{12}SS\alpha, \\
(MAX)_{21} = M_{12}^*A_{11}^0A_{11}^0 - M_{12}^*F_{A_{11}}A_{12}^0A_{21}^0A_{11}^0 + M_{22}F_SA_{21}^0A_{11}^0, \\
(MAX)_{22} = M_{12}^*F_{A_{11}}A_{12}^0A_{21}^0A_{11}^0 + M_{22}SS\alpha = M_{22}SS\alpha.
\]

Obviously, \((MAX)_{11} = (MAX)_{11}^*\), \((MAX)_{12} = (MAX)_{21}^*\) and \((MAX)_{22} = (MAX)_{22}^*\), i.e. \( MAX = (MAX)^* \).

On the other hand, let \( X \in A\{1, 3(M)\} \). Then the conditions (7) are satisfied and \( MAX = (MAX)^* \).

We have that \((MAX)_{21} = (MAX)_{21}^*\), so we obtain that \( M_{11}F_{A_{11}}A_{12}^0 = (M_{22}F_SA_{21}^0)^* \) and

\[
(M_{11}F_{A_{11}}A_{12}^0)(M_{11}F_{A_{11}}A_{12}^0)^* = M_{11}F_{A_{11}}A_{12}^0S\alpha M_{22}F_SA_{21}^0A_{11}^0 = M_{11}F_{A_{11}}A_{12}^0F_SA_{21}^0A_{11}^0 = 0.
\]

Hence, \( M_{11}F_{A_{11}}A_{12}^0 = 0 \), i.e. \( M_{11}F_{A_{11}}A_{12} = 0 \) and \( M_{22}F_SA_{21}^0A_{11}^0 = 0 \), i.e. \( M_{22}F_SA_{21} = 0 \).
Using the fact that $M$ is invertible, we have that $M_{11}$ and $M_{22}$ are also invertible, so we obtain the conditions (9).

By the $(MAX)_{11} = (MAX)^*_{11}$ and $(MAX)_{22} = (MAX)^*_{22}$, it follows that $A_{11}^\alpha \in A_{11}\{1,3(M_{11})\}$ and $S^\alpha \in S\{1,3(M_{22})\}$

The independence of the conditions (9) of the choice of $A_{11}^\alpha \in A_{11}\{1\}$ in $F_{A_{11}}$ and $S^\alpha \in S\{1\}$ in $F_S$, follows by the same arguments as in the proof of Theorem 2.1.

**Corollary 2.1** If $M$ is a positive definite matrix given by (6), such that

\[ S^\alpha M_{22}F_S = E_SM_{22}S^\alpha, \quad M_{12}^*FA_{11}A_{12} = 0, \quad M_{12}SS^\alpha = (M_{12}^*A_{11}A_{11}^\alpha)^*, \quad (10) \]

then $X \in A\{1,3(M)\}$ if and only if $A_{11}^\alpha \in A_{11}\{1,3(M_{11})\}$, $S^\alpha \in S\{1,3(M_{22})\}$ and

\[ F_{A_{11}}A_{12} = 0, \quad F_SA_{21} = 0. \quad (11) \]

The last two conditions are independent of the choice of $A_{11}^\alpha \in A_{11}\{1\}$ and $S^\alpha \in S\{1\}$ involved in $F_{A_{11}}$ and $F_S$.

Notice that for $M = I_{m+p}$, the conditions (8) are satisfied, so we obtain the Theorem 3 in [2] as a special case for $M = I_{m+p}$.

**Corollary 2.2** [2] $X \in A\{1,3\}$ if and only if $A_{11}^\alpha \in A_{11}\{1,3\}$, $S^\alpha \in S\{1,3\}$ and

\[ F_{A_{11}}A_{12} = 0, \quad F_SA_{21} = 0. \quad (12) \]

We have the following result with less restrictive conditions for the matrix $M$ only in one direction.

**Corollary 2.3** If $A_{11}^\alpha \in A_{11}\{1,3(M_{11})\}$, $S^\alpha \in S\{1,3(M_{22})\}$ and

\[ F_{A_{11}}A_{12} = 0, \quad F_SA_{21} = 0, \quad M_{12}SS^\alpha = (M_{12}^*A_{11}A_{11}^\alpha)^*, \]

then $X \in A\{1,3(M)\}$.

The following theorem give the necessary and sufficient conditions for $X \in A\{1,4(N)\}$.
Theorem 2.3 If $N$ is a positive definite matrix given by (6), such that

$$
(N_{12}S^\alpha S)^* = N_{12}^*A_{11}^\alpha A_{11} + N_{12}^*A_{11}^\alpha A_{12}S^\alpha A_{11},
$$

$$
S^\alpha M_{22}F_S = E_S M_{22}S^\alpha,
$$

then $X \in A\{1, \{4(N)\}\}$ if and only if $A_{11}^\alpha \in A_{11}\{1, \{4(N_{11})\}\}$, $S^\alpha \in S\{1, \{4(N_{22})\}\}$ and

$$
A_{12}E_S = 0, \ A_{21}E_{A_{11}} = 0.
$$

Proof. The proof is analogous to the proof of Theorem 2.2.

Corollary 2.4 If $N$ is nonnegative matrix given by (6), such that

$$
S^\alpha N_{22}F_S = E_S N_{22}S^\alpha, \ N_{12}^*A_{11}^\alpha A_{12} = 0, \ N_{12}S^\alpha S = (N_{12}^*A_{11}^\alpha A_{11})^*,
$$

then $X \in A\{1, \{4(N)\}\}$ if and only if $A_{11}^\alpha \in A_{11}\{1, \{4(N_{11})\}\}$, $S^\alpha \in S\{1, \{4(N_{22})\}\}$ and

$$
A_{12}E_S = 0, \ A_{21}E_{A_{11}} = 0.
$$

Also, Theorem 4 in [2] is obtained as a special case for $N = I_{n+q}$.

Corollary 2.5 [2] $X \in A\{1, \{4\}\}$ if and only if $A_{11}^\alpha \in A_{11}\{1, \{4\}\}$, $S^\alpha \in S\{1, \{4\}\}$ and

$$
A_{12}E_S = 0, \ A_{21}E_{A_{11}} = 0.
$$

Analogously to the Corollary 2.3 we have the following

Corollary 2.6 If $A_{11}^\alpha \in A_{11}\{1, \{4(N_{11})\}\}$, $S^\alpha \in S\{1, \{4(N_{22})\}\}$ and

$$
A_{12}E_S = 0, \ A_{21}E_{A_{11}} = 0. \ N_{12}S^\alpha S = (N_{12}^*A_{11}^\alpha A_{11})^*,
$$

then $X \in A\{1, \{3\}(M)\}$.

It is easy to see that Theorem 2.2 and Theorem 2.3 are satisfied if we suppose that $M_{11}$, $M_{22}$, $N_{11}$ and $N_{22}$ are invertible, instead of the fact that $M$ and $N$ are invertible matrices.

Using the results from Theorem 2.2, Theorem 2.3 and Theorem 2 in [2] we obtain the necessary and sufficient conditions such that the weighted Moore-Penrose inverse of $A$, $A_{M,N}^\dagger$ has the Banachiewicz form, where $M$ and $N$ are matrices which satisfy the conditions (8) and (13).
Theorem 2.4 Let $M$ and $N$ be the matrices which satisfy the conditions (8) and (13). Then $X = A^\dagger_{M,N}$ if and only if $A_{11}^\alpha = (A_{11})^\dagger_{M_{11},N_{11}}$, $S^\alpha = S^\dagger_{M_{22},N_{22}}$ and

$$F_{A_{11}}A_{12} = 0, \quad F_SA_{21} = 0, \quad A_{12}E_S = 0, \quad A_{21}E_{A_{11}} = 0. \quad (17)$$

If $M = I_{m+p}$ and $N = I_{n+q}$, then the conditions (8) and (13) are obviously satisfied, so from Theorem 2.4 we obtain the necessary and sufficient conditions for $X = A^\dagger$. Also, with less restrictive conditions for the matrices $M$ and $N$ we obtain the sufficient conditions for $X = A^\dagger_{M,N}$.

Corollary 2.7 If $A_{11}^\alpha = (A_{11})^\dagger_{M_{11},N_{11}}$, $S^\alpha = S^\dagger_{M_{22},N_{22}}$ and

$$F_{A_{11}}A_{12} = 0, \quad F_SA_{21} = 0, \quad A_{12}E_S = 0, \quad A_{21}E_{A_{11}} = 0,$$

$$M_{12}SS^\alpha = (M_{12}A_{11}A_{11}^\alpha)^*, \quad N_{12}S^\alpha S = (N_{12}A_{11}^\alpha A_{11})^*,$$

then $X = A^\dagger_{M,N}$.

It is interesting to notice that if we denote by

$$G = \begin{pmatrix} I & O \\ A_{21}A_{11}^\alpha & I \end{pmatrix} \begin{pmatrix} A_{11} & O \\ O & S \end{pmatrix} \begin{pmatrix} I & A_{11}^\alpha A_{12} \\ O & I \end{pmatrix},$$

where $A_{11}^\alpha \in A_{11}\{1\}$, then

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} O & F_{A_{11}}A_{12} \\ A_{21}E_{A_{11}} & O \end{pmatrix} + G,$$

and if the expression (5) of $X$ is rewritten as following matrices product

$$X = \begin{pmatrix} I & -A_{11}^\alpha A_{12} \\ O & I \end{pmatrix} \begin{pmatrix} A_{11}^\alpha & O \\ O & S^\alpha \end{pmatrix} \begin{pmatrix} I & 0 \\ -A_{21}A_{11}^\alpha & I \end{pmatrix},$$

then it is easy to see that $X \in G\{1\}$. Moreover, if the conditions $F_{A_{11}}A_{12} = 0$ and $A_{21}E_{A_{11}} = 0$ hold, we can obtain that $A = G$ and therefore $X \in A\{1,2\}$ if and only if $A_{11}^\alpha \in A\{1,2\}$ and $S^\alpha \in S\{1,2\}$.

In the rest of the paper we consider the sufficient conditions such that the W-weighted Drazin inverse can be represented in the Banachewiecz-Schur form.
Recall that for an arbitrary matrix \( W \in C^{(n+q)\times(m+p)} \) there exist non-singular matrices \( P \in C^{(n+q)\times(n+q)} \) and \( Q \in C^{(m+p)\times(m+p)} \) such that
\[
W = PW'Q^{-1} = P \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} Q^{-1},
\]
where \( W_1 \in C^{n\times m}, W_2 \in C^{q\times p} \).

Hence, if \( A = QA'P^{-1} \) and \( X = QX'P^{-1} \), then \( X \) is the \( W \)-weighted Drazin inverse of \( A \) if and only if \( X' \) is the \( W' \)-weighted Drazin inverse of \( A' \). With this reason we will naturally assume that \( W \in C^{(n+q)\times(m+p)} \) has the following form
\[
W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}, \quad W_1 \in C^{n\times m}, W_2 \in C^{q\times p}
\]
in the next result concerning \( W \)-weighted Drazin inverse. Furthermore, we will consider matrix \( A \in C^{(m+p)\times(n+q)} \) given by (3), modified Schur complement given by
\[
S = S(A) = A_{12} - A_{21}W_1A_{11}^\alpha W_1A_{12},
\]
and modified Banachiewicz-Schur form \( X \in C^{(n+q)\times(m+p)} \) given by
\[
X = \begin{bmatrix} A_{11}^\alpha + A_{11}^\alpha W_1A_{12}W_2S^\alpha W_2A_{21}W_1A_{11}^\alpha & -A_{11}^\alpha W_1A_{12}W_2S^\alpha \\ -S^\alpha W_2A_{21}W_1A_{11}^\alpha & S^\alpha \end{bmatrix},
\]
where \( A_{11}^\alpha = A_{11}^{dW_1}, S^\alpha = S^{dW_2} \).

**Theorem 2.5** Let \( A, X, W, S \) be given by (3), (20), (18), (19) respectively. If
\[
\begin{align*}
A_{12}W_2 &= A_{11}W_1A_{11}^\alpha W_1A_{12}W_2 = A_{12}W_2SW_2S^\alpha W_2, \\
A_{21}W_1 &= A_{21}W_1A_{11}W_1A_{11}^\alpha W_1 = SW_2S^\alpha W_2A_{21}W_1, \\
W_1A_{12} &= W_1A_{12}W_2S^\alpha W_2, \\
W_2A_{21} &= W_2A_{21}W_1A_{11}^\alpha W_1A_{11}, \\
A_{22}W_2 &= A_{22}W_2SW_2S^\alpha W_2,
\end{align*}
\]
then \( X = A^{dW} \).
Proof. By a straightforward computation, we obtain that

\[(AWX)_{11} = A_{11}W_1A_{11}^\alpha - (A_{12}W_2 - A_{11}W_1A_{11}^\alpha W_1A_{12}W_2)S^\alpha W_2A_{21}W_1A_{11}^\alpha \]
\[= A_{11}W_1A_{11}^\alpha, \text{ using the first part of (21),} \]
\[(AWX)_{12} = (A_{12}W_2 - A_{11}W_1A_{11}^\alpha W_1A_{12}W_2)S^\alpha \]
\[= 0, \text{ by the first the part of (21),} \]
\[(AWX)_{21} = A_{21}W_1A_{11}^\alpha - (A_{22} - A_{21}W_1A_{11}^\alpha W_1A_{12})W_2S^\alpha W_2A_{21}W_1A_{11}^\alpha \]
\[= A_{21}W_1A_{11}^\alpha - SW_2S^\alpha W_2A_{21}W_1A_{11}^\alpha \]
\[= 0, \text{ by the second part of (22),} \]
\[(AWX)_{22} = (A_{22} - A_{21}W_1A_{11}^\alpha W_1A_{12})W_2S^\alpha = SW_2S^\alpha. \]

Similarly,

\[(XWA)_{11} = A_{11}^\alpha W_1A_{11} - A_{11}^\alpha W_1A_{11}W_2S^\alpha (W_2A_{21} - W_2A_{21}W_1A_{11}^\alpha W_1A_{11}) \]
\[= A_{11}^\alpha W_1A_{11}, \text{ using (24),} \]
\[(XWA)_{12} = A_{11}^\alpha W_1A_{12} - A_{11}^\alpha W_1A_{12}W_2S^\alpha W_2 (A_{22} - A_{21}W_1A_{11}^\alpha W_1A_{12}) \]
\[= A_{11}^\alpha W_1A_{12} - A_{11}^\alpha W_1A_{12}W_2S^\alpha W_2S \]
\[= 0, \text{ using (23),} \]
\[(XWA)_{21} = S^\alpha (W_2A_{21} - W_2A_{21}W_1A_{11}^\alpha W_1A_{11}) \]
\[= 0, \text{ by (24),} \]
\[(XWA)_{22} = S^\alpha W_2(A_{22} - A_{21}W_1A_{11}^\alpha W_1A_{12}) \]
\[= SW_2S^\alpha. \]

Now,

\[AWX = \begin{bmatrix} A_{11}W_1A_{11}^\alpha & 0 \\ 0 & SW_2S^\alpha \end{bmatrix} \]

and

\[XWA = \begin{bmatrix} A_{11}^\alpha W_1A_{11} & 0 \\ 0 & S^\alpha W_2S \end{bmatrix}, \]

so \(AWX = XWA.\)

Using the facts that \(A_{11}^\alpha = A_{11}^{d,W_1}\) and \(S^\alpha = S^{d,W_2}\), we obtain that \(XWA = X\). Also, using (25),(22) and (21) it follows that,

\[(AW)^2XW = \begin{bmatrix} (A_{11}W_1)^2A_{11}^\alpha W_1 & A_{12}W_2SW_2S^\alpha W_2 \\ A_{21}W_1A_{11}W_1A_{11}^\alpha W_1 & A_{22}W_2SW_2S^\alpha W_2 \end{bmatrix} \]
\[= AW + \begin{bmatrix} (A_{11}W_1)^2A_{11}^\alpha W_1 - A_{11}W_1 & 0 \\ 0 & 0 \end{bmatrix}. \]
By the induction, using the first part of (22), we obtain that

$$(AW)^{m+1}XW = (AW)^m + \left[ \begin{array}{cc} (A_{11}W_1)^m - (A_{11}W_1)^{m+1}A_{11}^0W_1 & 0 \\ 0 & 0 \end{array} \right].$$

Hence $(AW)^{m+1}XW = (AW)^m$, for an arbitrary $m \geq \text{ind}(A_{11}W_1)$.

From Theorem 2.5, we obtain the result of Wei [Theorem 1, [15]] when $m = n$, $p = q$ and $W = I_{m+p}$:

**Corollary 2.8** Let $A$ and $X$ are given by (3) and (5). If $F_{A_{11}A_{12}} = 0$, $A_{21}E_{A_{11}} = 0$, $A_{12}E_S = 0$, $E_SA_{21} = 0$, $A_{22}E_S = 0$, then $X = A^d$.

### 3 Conclusions

In this paper we developed some necessary and sufficient conditions for which the weighted generalized inverses $A^{(1,3M)}$, $A^{(1,4N)}$, and $A^{M,N}_{M,N}$ of a partitioned matrix can be respectively expressed in Banachiewicz-Schur form under certain assumptions. Similar problem for the $W$-weighted Drazin inverse is also considered and some sufficient conditions are established. This can be viewed as a generalization of the earlier works obtained bt S.K.Baksalary and Styan [2] and Y.Wei [15].

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**References**


