Representations of generalized inverses of partitioned matrix involving Schur complement *

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Abstract

In this article, we consider some representations of \( f_{1,3} \); \( f_{1,4} \); \( f_{1,2,3} \) and \( f_{1,2,4} \)-inverses of a partitioned matrix \( M \) which are equivalent to some rank additivity conditions. We present some applications of these results to generalizations of the Sherman-Morrison-Woodbury-type formulae.

Keywords: Partitioned matrix; Rank additivity conditions; Sherman-Morrison-Woodbury formula


1 Introduction

Let \( \mathbb{C}^{m \times n} \) denote the set of all \( m \times n \) complex matrices. We use \( r(A) \) and \( A^* \) to denote the rank and the conjugate transpose matrix of a matrix \( A \), respectively. The Moore-Penrose inverse of a matrix \( A \in \mathbb{C}^{m \times n} \) is a matrix \( X \in \mathbb{C}^{n \times m} \) which satisfies

\[
\begin{align*}
(1) \quad AXA &= A \\
(2) \quad XAX &= X \\
(3) \quad (AX)^* &= AX \\
(4) \quad (XA)^* &= XA.
\end{align*}
\]

The Moore-Penrose inverse of \( A \) is unique and it is denoted by \( A^\dagger \).

For any \( A \in \mathbb{C}^{m \times n} \), let \( A\{i, j, \ldots, k\} \) denote the set of all \( X \in \mathbb{C}^{n \times m} \) which satisfy equations \((i), (j), \ldots, (k)\) of (1.1). In this case \( X \) is a \( \{i, j, \ldots, k\} \)-inverse of \( A \) which we denote by \( A^{(i, j, \ldots, k)} \). Evidently, \( A\{1, 2, 3, 4\} = \{A^\dagger\} \).

For a complex matrix \( M \) of the form

\[
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{C}^{(m+s) \times (n+t)}, \quad A \in \mathbb{C}^{m \times n}
\]

\[
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{C}^{(m+s) \times (n+t)}, \quad A \in \mathbb{C}^{m \times n}
\]

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in the case when \( m = n \) and \( A \) is invertible, the Schur complement of \( A \) in \( M \) is defined by \( S_A = D - CA^{-1}B \). Sometimes, we denote the Schur complement of \( A \) in \( M \) by \( (M/A) \). Similarly, if \( s = t \) and \( D \) is invertible, then the Schur complement of \( D \) in \( M \) is defined by \( S_D = A - BD^{-1}C \).

In the case when \( A \) is not invertible, for a matrix given by (1.2) and any fixed generalized inverse \( A^* \in A\{1\} \), the generalized Schur complement of \( A \) in \( M \) is defined by

\[
S_A = D - CA^{-1}B.
\]

Similarly, for some fixed \( D^* \in D\{1\} \), the generalized Schur complement of \( D \) in \( M \) is defined by

\[
S_D = A - BD^{-1}C.
\]

The Schur complement and generalized Schur complement have quite important applications in the matrix theory, statistics, numerical analysis, applied mathematics, etc. It is well-known that if \( A \in \mathbb{C}^{m \times m} \) is nonsingular then the invertibility of the matrix \( M \) is equivalent to the invertibility of the Schur complement of \( A \) in \( M \) and in that case the inverse of \( M \) is given by

\[
M^{-1} = \begin{pmatrix}
A^{-1} + A^{-1}BS_A^{-1}CA^{-1} & -A^{-1}BS_A^{-1} \\
-S_A^{-1}CA^{-1} & S_A^{-1}
\end{pmatrix}.
\] (1.3)

The expression (1.3) is called the Banachiewicz-Schur form of the matrix \( M \). Some variations of (1.3) can be found in [23]. Equation (1.3) is one of the most useful formulae in matrix theory and its applications, and was extensively applied to manipulate various operations related to partitioned matrices and their inverses. If, however, the matrix \( A \) is singular, then (1.3) no longer holds. In this case, we have to use generalized inverses of the submatrices in \( M \) to construct a partitioned expression of the inverse of \( M \). In an earlier paper of Tian [28] on the Moore-Penrose inverses of partitioned matrices under rank additivity conditions, the following formula for the inverse of \( M \) is presented:

\[
M^t = \begin{pmatrix}
K_1 - K_2CA^t - A^tB\bar{K}_3 + A^tB\bar{J}_DCA^t & K_2 - A^tB\bar{J}_D \\
K_3 - J_DCA^t & \bar{J}_D
\end{pmatrix},
\]

where

\[
E_X = I - X^tX, \quad F_X = I - XX^t, \\
S_A = D - CA^tB, \quad B_1 = F_AB, \quad C_1 = CE_A, \quad J_D = F_{C_1}S_AE_{B_1}, \\
K_1 = A^t + C_1[I(S_AJ_D^tS_A - S_A)B_1^t], \quad K_2 = C_1[I - S_AJ_D^t], \\
K_3 = (I - J_D^tS_A)B_1^t.
\]

Partitioned matrices are very useful in investigating various properties of generalized inverses and hence can be widely used in the matrix theory and have many other
applications (see [16, 17, 18, 24, 26, 27]). Various forms of generalized inverses of the partitioned matrix $M$ in the case when it is not invertible have been discovered in many recent papers. Some results concerning the expressions for generalized inverses of $M$ in the special case when $D = 0$ under the following two special rank additivity conditions:

$$r(M) = r \left( \begin{array}{c} A \\ C \end{array} \right) + r(B) = r(A, B) + r(C)$$

and

$$r(M) = r(A) + r(B) + r(C)$$

can be found in the papers of S.K. Mitra[24] and Y. Chen et al. [6]. S.K. Mitra[24] proved that if (1.4) holds, then

$$M^\dagger = \left( \begin{array}{ccc} L^\dagger & C^\dagger - L^\dagger AC^\dagger \\ B^\dagger - B^\dagger AL^\dagger & B^\dagger (AL^\dagger A - A)C^\dagger \end{array} \right),$$

where $L = F_B AE_C$.

Y. Chen [7] proved that the rank additivity condition

$$r(M) = r \left( \begin{array}{c} A \\ C \end{array} \right) + r \left( \begin{array}{c} B \\ D \end{array} \right) = r(A, B) + r(C, D)$$

is necessary and sufficient for the block independence in g-inverse of the partitioned matrix $M$ and for the Moore-Penrose inverse $M^\dagger$ to have some special forms. G.Wei et al. [14] and M.Wei et al. [33] studied structures of least squares g-inverses and minimum norm g-inverses of a bordered matrix by using QQ-SVD. Y.Tian [29] showed some rank equalities for the $2 \times 2$ partitioned matrix $M$ and presented an expression of the Moore-Penrose inverse of $M$ under the rank additivity condition (1.5).

In this paper, we will further investigate the relationship between the expressions given by Tian [28] and the rank additivity conditions. In Section 2, we give some necessary and sufficient rank additivity conditions for $\{1, 3\}, \{1, 4\}, \{1, 2, 3\}$ and $\{1, 2, 4\}$-inverses of the partitioned matrix $M$ to be presented in appropriate forms. Also, we establish necessary and sufficient conditions for the Moore-Penrose inverse of $M$ to assume some special forms. As a corollary, we get some results obtained by Y.Chen [7]. Also, almost all the results from Section 2 generalize some of the results from [28]. In Section 3, we present an application of these results to generalizations of the Sherman-Morrison-Woodbury-type formulae.

Some well-known rank equalities for partitioned matrices are given below.

**Lemma 1.1.** [7] Let $M$ be given by (1.2) and let $B_1 = FA, C_1 = CE, J_D = FC_1 S_A E B_1, S_A = D - CA^{(1)} B$. Then:

(a) The following statements are equivalent:

(i) $r(M) = r \left( \begin{array}{c} A \\ C \end{array} \right) + r \left( \begin{array}{c} B \\ D \end{array} \right)$.
Lemma 1.2. \[34\] Let \( A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{m \times t}, C \in \mathbb{C}^{s \times n}, \) and \( D \in \mathbb{C}^{s \times t} \). Then

\[(a) \quad r\left( \begin{array}{cc} A & B \\ C & 0 \end{array} \right) = r(A) + r(B) + r(C) \]

if and only if

\[R(A) \cap R(B) = \{0\}, \quad R(A^*) \cap R(C^*) = \{0\}.\]

\[(b) \quad r\left( \begin{array}{cc} 0 & B \\ C & D \end{array} \right) = r(B) + r(C) + r(D) \]

if and only if

\[R(C) \cap R(D) = \{0\}, \quad R(B^*) \cap R(D^*) = \{0\}.\]

Lemma 1.3. \[18\] Let \( A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{m \times t}, C \in \mathbb{C}^{s \times n}, \) then

\[(a) \quad \] The following statements are equivalent:
\[(i)\] \(r(A + B) = r(A) + r(B)\);
\[(ii)\] \(R(A) \cap R(B) = \{0\}\);
\[(iii)\] \((FAB)^\dagger(FAB) = B^\dagger B\).

(b) The following statements are equivalent:

\[(i)\] \(r\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right) = r(A) + r(C)\);
\[(ii)\] \(R(A) \cap R(C^\ast) = \{0\}\);
\[(iii)\] \((CE_A)(CE_A)^\dagger = CC^\dagger\).

2 Main results

In this section, we give rank additivity necessary and sufficient conditions to present \(\{1,3\}, \{1,4\}, \{1,2,3\}\) and \(\{1,2,4\}\)-inverses of the partitioned matrix \(M\) in appropriate forms. Also, we establish some necessary and sufficient conditions for the Moore-Penrose inverse of \(M\) to assume some special forms.

For convenience, we first state some notations which will be helpful throughout the paper:

\[
S_A = D - CA^{-1}B, \quad B_1 = F_AB, \quad C_1 = CE_A, \quad J_D = F_{C_1}S_AE_{B_1},
K_1 = A^{-1} + C_1^{-1}(S_AJ_D^1S_A - S_A)B_1^{-1},
K_2 = C_1^{-1}(I - S_AJ_D^1),
K_3 = (I - J_D^1S_A)B_1^{-1},
\]

where \(A^{-1}, C_1^{-1}, B_1^{-1}\) are arbitrary inner inverses of \(A, C_1, B_1\), respectively.

Using that \(X^\dagger = (X^\ast X)^\dagger X^\ast\), \(X^\dagger = X^\ast(XX^\dagger)^\dagger\) and (2.1), it is easy to verify

\[
A^\dagger B_1 = 0, \quad AC_1^\dagger = 0, \quad C_1A^\dagger = 0, \quad B_1^\dagger A = 0, \quad B_1J_D^1 = 0, \quad J_D^1 C_1 = 0.
\]

Also, using that \((PAQ)^\dagger = Q(PAQ)^\dagger = (PAQ)^\dagger P = Q(PAQ)^\dagger P\) in the case when \(P, Q\) are Hermitian idempotents, we get that

\[
E_{B_1}J_D^1 = J_D^1 F_{C_1} = E_{B_1}J_D^1 F_{C_1} = J_D^1,
F_{C_1}(S_A - S_AJ_D^1S_A)E_{B_1} = F_{C_1}(S_A - S_AE_{B_1}J_D^1S_AF_{C_1})E_{B_1} = 0.
\]

**Theorem 2.1.** Let \(M\) be given by (1.2). If \(X\) is defined by

\[
X = \begin{pmatrix} K_1 - K_2CA^{-1} - A^{-1}BK_3 + A^{-1}BJ_D^1CA^{-1} & K_2 - A^{-1}BJ_D^1 \\ K_3 - J_D^1CA^{-1} & J_D^1 \end{pmatrix}
\]

then the following hold:
(a) If we take that $C_1^\top = C_1^\top$, then for any $A^\top \in A\{1,3\}$, $B_1^\top \in B_1\{1,3\}$, we have that $X \in M\{1,3\}$ if and only if
\[ r(M) = r(A, B) + r(C, D). \]

(b) If we take that $B_1^\top = B_1^\top$, then for any $A^\top \in A\{1,4\}$, $C_1^\top \in C_1\{1,4\}$, we have that $X \in M\{1,4\}$ if and only if
\[ r(M) = r\left( \begin{pmatrix} A \\ C \end{pmatrix} \right) + r\left( \begin{pmatrix} B \\ D \end{pmatrix} \right). \]

Proof. (a)$\implies$) A simple computation shows that
\[
MX = \begin{pmatrix}
AA^\top + B_1B_1^\top & 0 \\
(I - J_DJ_D^\top)(I - C_1C_1^\top)(CA^\top + SB_1^\top) & J_DJ_D^\top + C_1C_1^\top
\end{pmatrix}.
\]
Since $X \in M\{1,3\}$, it follows that
\[
(I - J_DJ_D^\top)(I - C_1C_1^\top)(CA^\top + SB_1^\top) = 0.
\]
Since $MXM = M$ is equivalent to
\[
\begin{pmatrix}
A + B_1B_1^\top A & AA^\top B + B_1B_1^\top B \\
(J_DJ_D^\top + C_1C_1^\top)C & (J_DJ_D^\top + C_1C_1^\top)D
\end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]
we get
\[
(J_DJ_D^\top + C_1C_1^\top)C = C, \quad (J_DJ_D^\top + C_1C_1^\top)D = D,
\]
i.e. using that $J_D^\top C_1 = 0$, we have
\[
(I - J_DJ_D^\top)(I - C_1C_1^\top)C = 0, \quad (I - J_DJ_D^\top)(I - C_1C_1^\top)D = 0.
\]

Thus, by Lemma 1.1 (b), we conclude that
\[ r(M) = r(A, B) + r(C, D). \]

$\Leftarrow$) By Lemma 1.1 (b), it follows that
\[
(I - J_DJ_D^\top)(I - C_1C_1^\top)C = 0, \quad (I - J_DJ_D^\top)(I - C_1C_1^\top)S_A = 0
\]
Now, it is easy to check that Penrose equations $MXM = M$ and $MX = (MX)^*$ are satisfied.

(b) The case $\{1,4\}$ is treated completely analogously, and the corresponding result follows by taking transpose conjugate.

Remark that if we apply Theorem 2.1 for the matrix $M_1 = \begin{pmatrix} D & C \\ B & A \end{pmatrix} = PMP$, where $P = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ and use that $PXP \in M\{1,3\} \Leftrightarrow X \in M_1\{1,3\}$, we get the following result:
Theorem 2.2. Let $M$ be given by (1.2). If $X$ is defined by

$$X = \begin{pmatrix} J_A^1 & L_2 - J_A^1 BD^- \\ L_3 - D^- CJ_A^1 & L_1 - L_3 BD^- - D^- CL_2 + D^- CJ_A^1 BD^- \end{pmatrix},$$

(2.5)

then the following hold:

(a) If we take that $B_1^- = B_1^+$, then for any $D^- \in D\{1,3\}$, $C_1^- \in C_1\{1,3\}$, we have that $X \in M\{1,3\}$ if and only if

$$r(M) = r(A, \ B) + r(C, \ D).$$

(b) If we take that $C_1^- = C_1^+$, then for any $D^- \in D\{1,4\}$, $B_1^- \in B_1\{1,4\}$, we have that $X \in M\{1,4\}$ if and only if

$$r(M) = r\left( \begin{array}{c} A \\ C \end{array} \right) + r\left( \begin{array}{c} B \\ D \end{array} \right),$$

where

$$C_1 = F_D C, \ B_1 = B ED, \ J_A = F_B S_D E C_1, \ L_1 = D^- + B_1^- (S_D J_A^1 S_D - S_D) C_1^-, \ L_2 = (I - J_A^1 S_D) C_1^-, \ L_3 = B_1^- (I - S_D J_A^1).$$

In a similar way Theorem 2.2 has just been derived from Theorem 2.1 we can also deduce appropriate results with each of the forthcoming theorems of this section playing the role of Theorem 2.1.

In the following theorem, we will show that the same conditions as in Theorem 2.1 are necessary and sufficient for the case $\{1,2,3\}$ and for the case $\{1,2,4\}$.

Theorem 2.3. Let $M$ be given by (1.2). If $X$ is defined by (2.4), then the following hold:

(a) If we take that $C_1^- = C_1^+$, then for any $A^- \in A\{1,2,3\}$ and any $B_1^- \in B_1\{1,2,3\}$, we have that $X \in M\{1,2,3\}$ if and only if

$$r(M) = r(A, \ B) + r(C, \ D).$$

(b) If we take that $B_1^- = B_1^+$, then for any $A^- \in A\{1,2,4\}$ and any $C_1^- \in C_1\{1,2,4\}$, we have that $X \in M\{1,2,4\}$ if and only if

$$r(M) = r\left( \begin{array}{c} A \\ C \end{array} \right) + r\left( \begin{array}{c} B \\ D \end{array} \right).$$

(c) [6, 28] If we take that $A^- = A^+$, $B_1^- = B_1^+$, $C_1^- = C_1^+$, then $X = M^+$ if and only if

$$r(M) = r(A, \ B) + r(C, \ D) = r\left( \begin{array}{c} A \\ C \end{array} \right) + r\left( \begin{array}{c} B \\ D \end{array} \right).$$

(2.6)
Proof. (a) \((\Leftrightarrow)\) By Theorem 2.1 (a), we conclude that \(X \in M \{1, 3\}\). By (2.2) and (2.3), we have

\[
MX = \begin{pmatrix}
AA^{-} + B_1 B_1^\dagger & 0 \\
0 & J_D J_D^\dagger + C_1 C_1^\dagger
\end{pmatrix},
\]

which implies that \(XMX = X\). The reverse part of the proof follows obviously.

(b) The proof follows from item (a) by taking transpose conjugate.

(c) The result follows by (a) and (b).

The next theorem is the converse of a result given in [28], where the author proved that under the same conditions the representation of \(M^\dagger\) below is valid, i.e. we prove that these conditions are also necessary that \(M^\dagger\) has representation (2.7). We give the proofs of both directions of the assertion because our proof of the part given in [28] differs from the one presented there.

**Theorem 2.4.** Let \(M\) be given by (1.2). Then

\[
M^\dagger = \begin{pmatrix}
A^\dagger - K_2 C A^\dagger - A^\dagger B K_3 + A^\dagger B J_D^\dagger C A^\dagger & K_2 - A^\dagger B J_D^\dagger \\
K_3 - J_D^\dagger C A^\dagger & J_D^\dagger
\end{pmatrix}
\] (2.7)

if and only if

\[
\begin{align*}
R\left( \begin{pmatrix} A \\ 0 \end{pmatrix} \right) & \subseteq R(M), \quad R\left( \begin{pmatrix} A^* \\ 0 \end{pmatrix} \right) \subseteq R(M^*), \\
R(B_1^\dagger) \cap R(S_A^\dagger) & = \{0\}, \quad R(C_1) \cap R(S_A) = \{0\},
\end{align*}
\]

where \(S_A = D - C A^\dagger B, B_1 = F_A B, C_1 = C E_A, J_D = F_C S_A E_B, K_2 = C_1^\dagger (I - S_A J_D^\dagger), K_3 = (I - J_D^\dagger S_A) B_1^\dagger\).

**Proof.** (\(\Rightarrow\)) Suppose that \(M^\dagger\) is given by (2.7). A simple computation shows that

\[
MM^\dagger = \begin{pmatrix}
AA^\dagger + B_1 B_1^\dagger & 0 \\
(I - J_D J_D^\dagger - C_1 C_1^\dagger) C A^\dagger + (S_A - S_A J_D^\dagger S_A) B_1^\dagger & J_D J_D^\dagger + C_1 C_1^\dagger
\end{pmatrix}.
\]

Hence,

\[
(I - J_D J_D^\dagger - C_1 C_1^\dagger) C A^\dagger + (S_A - S_A J_D^\dagger S_A) B_1^\dagger = 0.
\] (2.8)

Multiplying (2.8) by \(A\) from the right side and using that \(B_1^\dagger A = 0\), we get

\[
(I - J_D J_D^\dagger - C_1 C_1^\dagger) C A^\dagger A = 0.
\]

Since \((I - J_D J_D^\dagger - C_1 C_1^\dagger) C_1 = 0\), the last equality implies that

\[
(I - J_D J_D^\dagger - C_1 C_1^\dagger) C = 0.
\]

Now by (2.8), we have

\[
(S_A - S_A J_D^\dagger S_A) B_1^\dagger = 0,
\]
which implies that $K_1 = A^1$. Now, by Theorem 2.3 (c) we have that rank additivity condition (2.6) is satisfied which by Lemma 1.5 [28] implies that

$$R\left( \begin{array}{c} A \\ 0 \end{array} \right) \subseteq R(M), R\left( \begin{array}{c} A^* \\ 0 \end{array} \right) \subseteq R(M^*).$$

Analogously, by $M^tM = (M^tM)^*$, we get

$$B(I - B_1^t B_1 - J_D^t J_D) = 0, \quad C_1^t (S_A - S_A J_D^t S_A) = 0.$$ 

Since $F_1(S_A - S_A J_D S_A) E_{B_1} = 0$, by

$$(S_A - S_A J_D S_A) B_1^t = 0, \quad C_1^t (S_A - S_A J_D S_A) = 0,$$

we get $S_A J_D S_A = S_A$. By (2.3) we get

$$J_D^t S_A J_D^t = J_D^t F_1 S_A E_{B_1} J_D^t = J_D^t J_D^t J_D^t = J_D^t$$

which implies that $J_D^t \in S_A \{1, 2\}$. Thus, $r(J_D^t) = r(J_D) = r(S_A)$. Therefore,

$$r\left( \begin{array}{c} 0 \\ C_1 \\ S_A \end{array} \right) = r(B_1) + r(C_1) + r(F_1 S_A E_{B_1}) = r(B_1) + r(C_1) + r(S_A).$$

By Lemma 1.2, we conclude that

$$R(B_1^t) \cap R(S_A^*) = \{0\}, \quad R(C_1) \cap R(S_A) = \{0\}.$$

$(\Leftarrow)$ By

$$R(B_1^t) \cap R(S_A^*) = \{0\}, \quad R(C_1) \cap R(S_A) = \{0\},$$

from Lemma 1.2 (b), we have that

$$r\left( \begin{array}{c} 0 \\ C_1 \\ S_A \end{array} \right) = r(B_1) + r(C_1) + r(S_A). \quad (2.9)$$

Since $r\left( \begin{array}{c} 0 \\ C_1 \\ S_A \end{array} \right) = r(B_1) + r(C_1) + r(J_D)$, we have that $r(J_D) = r(S_A)$ which together with $J_D^t = J_D^t S_A J_D^t$ using Theorem 2 [1] (page 45) gives that $J_D^t \in S_A \{1\}$, i.e. $S_A = S_A J_D^t S_A$. Now

$$r(S_A E_{B_1}) \leq r(S_A) = r(J_D) \leq r(S_A E_{B_1})$$

i.e. $r(S_A E_{B_1}) = r(S_A)$, which implies that

$$r\left( \begin{array}{c} B_1 \\ S_A \end{array} \right) = r(B_1) + r(S_A E_{B_1}) = r(B_1) + r(S_A). \quad (2.10)$$

Now, by (2.9) and (2.10), we have

$$r\left( \begin{array}{c} 0 \\ C_1 \\ S_A \end{array} \right) = r\left( \begin{array}{c} B_1 \\ S_A \end{array} \right) + r(C_1).$$
Similarly, we get that
\[
  r\begin{pmatrix} 0 & B_1 \\ C_1 & S_A \end{pmatrix} = r(B_1) + r(C_1, S_A).
\]

Using the last two equalities and that
\[
  R\begin{pmatrix} A \\ 0 \end{pmatrix} \subseteq R(M), R\begin{pmatrix} A^* \\ 0 \end{pmatrix} \subseteq R(M^*),
\]
by Lemma 1.5 [28], we get that the rank additivity condition (2.6) holds. Thus, by Theorem 2.3 (c), we get
\[
  M^\dagger = \begin{pmatrix} K_1 - K_2CA^\dagger - A^\dagger BK_3 + A^\dagger BJ_D^\dagger CA^\dagger & K_2 - A^\dagger BJ_D^\dagger \\ K_3 - J_D^\dagger CA^\dagger & J_D^\dagger \end{pmatrix}
\]
where \( K_1, K_2, K_3 \) are defined by (2.1) and \( A^- = A^\dagger, B_1^- = B_1^\dagger, C_1^- = C_1^\dagger \), which can be simplified to (2.7) since \( S_A = S_AJ_D^\dagger S_A \).

In Corollary 2.5 [28], the author proved that conditions given in the next theorem are sufficient for \( M^\dagger \) to be presented by (2.11). In the next theorem we prove that they are also necessary.

**Theorem 2.5.** Let \( M \) be given by (1.2). Then
\[
  M^\dagger = \begin{pmatrix} A^\dagger - C_1^\dagger CA^\dagger - A^\dagger BB_1^\dagger + A^\dagger BS_A^\dagger CA^\dagger & C_1^\dagger - A^\dagger BS_A^\dagger \\ B_1^\dagger - S_A^\dagger CA^\dagger & S_A^\dagger \end{pmatrix}
\]
if and only if
\[
  R\begin{pmatrix} A \\ 0 \end{pmatrix} \subseteq R(M), R\begin{pmatrix} A^* \\ 0 \end{pmatrix} \subseteq R(M^*),
\]
\[
  R(BS_A^\dagger) \subseteq R(A), R(C^*S_A) \subseteq R(A^*),
\]
where \( S_A = D - CA^\dagger B, \ B_1 = F_AB, \ C_1 = CE_A \).

**Proof.** \((\Rightarrow)\) Since \( AC_1^- = 0 \) and \( A^\dagger B_1 = 0 \), we get
\[
  MM^\dagger = \begin{pmatrix} AA^\dagger + B_1B_1^\dagger - B_1S_A^\dagger CA^\dagger & B_1S_A^\dagger \\ (I - S_AS_A^\dagger - C_1C_1^\dagger)CA^\dagger + S_AB_1^\dagger & S_A^\dagger S_A^\dagger + C_1^\dagger \end{pmatrix}
\]
and
\[
  M^\dagger M = \begin{pmatrix} A^\dagger A + C_1^\dagger C_1 - A^\dagger BS_A^\dagger C_1 & A^\dagger B(I - S_AS_A^\dagger - B_1^\dagger B_1) + C_1^\dagger S_A \\ S_A^\dagger C_1 & S_A^\dagger S_A^\dagger + B_1^\dagger B_1 \end{pmatrix}.
\]
Moreover, since \( MM^\dagger \) and \( M^\dagger M \) are Hermitian, we have from (2.12) and (2.13) that
\[
  AA^\dagger + B_1B_1^\dagger - B_1S_A^\dagger CA^\dagger = AA^\dagger + B_1B_1^\dagger - (B_1S_A^\dagger CA^\dagger)^*\tag{2.14}
\]
and
\[ A^\dagger A + C_1^\dagger C_1 - A^\dagger B S_A^\dagger C_1 = A^\dagger A + C_1^\dagger C_1 - (A^\dagger B S_A^\dagger C_1)^*. \tag{2.15} \]

By \( B_1^A A = 0, B_1^A = 0, AC_1^A = 0, AC_1^A = 0 \), multiplying (2.14) by \( A \) from the right side and (2.15) by \( A \) from the left side, we get

\[ B_1 S_A^\dagger C A^\dagger A = 0, \quad AA^\dagger B S_A^\dagger C_1 = 0, \]

that is

\[ B_1 S_A^\dagger C A^\dagger = 0, \quad A^\dagger B S_A^\dagger C_1 = 0. \]

Moreover, we have

\[ (I - S_A S_A^\dagger - C_1 C_1^\dagger) C A^\dagger + S_A B_1^\dagger = (B_1 S_A^\dagger)^*. \tag{2.16} \]

and

\[ A^\dagger B (I - S_A S_A^\dagger - B_1^B B_1) + C_1 S_A = (S_A S_A^\dagger C_1)^*. \tag{2.17} \]

Again multiplying (2.16) by \( A \) from the right side and (2.17) by \( A \) from the left side, we get

\[ (I - S_A S_A^\dagger - C_1 C_1^\dagger) C A^\dagger A = 0, \quad AA^\dagger B (I - S_A S_A^\dagger - B_1^B B_1) = 0, \]

i.e.

\[ (I - S_A S_A^\dagger - C_1 C_1^\dagger) C A^\dagger = 0, \quad A^\dagger B (I - S_A S_A^\dagger - B_1^B B_1) = 0. \]

Thus,

\[ MM^\dagger = \begin{pmatrix} AA^\dagger + B_1 B_1^\dagger & B_1 S_A^\dagger \\ S_A B_1^\dagger & S_A S_A^\dagger + C_1 C_1^\dagger \end{pmatrix} \]

and

\[ M^\dagger M = \begin{pmatrix} A^\dagger A + C_1^\dagger C_1 & C_1^\dagger S_A \\ S_A^\dagger C_1 & S_A^\dagger S_A + B_1^B B_1 \end{pmatrix}. \]

By \( (MM^\dagger)M = M \) and \( M(M^\dagger M) = M \) we get

\[ \begin{pmatrix} A + B_1 B_1^\dagger A + B_1 S_A^\dagger C \\ S_A B_1^\dagger A + (S_A S_A^\dagger + C_1 C_1^\dagger) C \end{pmatrix} = \begin{pmatrix} AA^\dagger B + B_1 B_1^\dagger B + B_1 S_A^\dagger D \\ S_A B_1^\dagger B + (S_A S_A^\dagger + C_1 C_1^\dagger) D \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.18} \]

and

\[ \begin{pmatrix} A + BS_A^\dagger C_1 \\ C A^\dagger A + CC_1^\dagger C_1 + DS_A^\dagger C_1 \end{pmatrix} = \begin{pmatrix} B(S_A S_A^\dagger + B_1^B B_1) \\ CC_1^\dagger S_A + D(S_A S_A^\dagger + B_1^B B_1) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{2.19} \]

By (2.18) and (2.2),

\[ B_1 S_A^\dagger C = 0, \tag{2.20} \]

\[ B_1 S_A^\dagger D = 0, \tag{2.21} \]

\[ S_A B_1^\dagger A + (S_A S_A^\dagger + C_1 C_1^\dagger) C = C, \]

\[ S_A B_1^\dagger A + (S_A S_A^\dagger + C_1 C_1^\dagger) C = C. \]
\[ S_A B_1^t B + (S_A S_A^t + C_1 C_1^t) D = D. \]

By (2.20) and (2.21), we get \( B_1 S_A^t S_A = 0 \), i.e. \( B_1 S_A^t = 0 \), which implies \( R(B_S A^t) \subseteq R(A) \). Also, \( B_1 S_A^t = 0 \) which together with \( B_1^t A = 0 \) by simple computation shows that

\[
MM^t \begin{pmatrix} A \\ 0 \end{pmatrix} = \begin{pmatrix} A \\ 0 \end{pmatrix},
\]

which is equivalent to

\[
R \begin{pmatrix} A \\ 0 \end{pmatrix} \subseteq R(M).
\]

Analogously, by (2.19), we get

\[
R(C^* S_A^t) \subseteq R(A^*), \quad R \begin{pmatrix} A^* \\ 0 \end{pmatrix} \subseteq R(M^*).
\]

\( (\Leftarrow) \) It follows by Corollary 2.5 [28]. \qed

**Theorem 2.6.** Let \( M \) be given by (1.2) and let \( S_A, B_1, C_1 \) be defined by (2.1). Let

\[
X = \begin{pmatrix} A^- - C_1^- C A^- - A^- B B_1^- - C_1^- S A B_1^- & C_1^- \\ B_1^- & 0 \end{pmatrix}.
\]

Then the following hold:

(a) If we take that \( C_1^- = C_1^t \), then for any \( A^- \in A \{1, 3\} \) and any \( B_1^- \in B_1 \{1, 3\} \), we have that \( X \in M \{1, 3\} \) if and only if

\[
R(A^*) \cap R(C^*) = \{0\}, \quad R(D) \subseteq R(C).
\]

(b) If we take that \( B_1^- = B_1^t \), then for any \( A^- \in A \{1, 4\} \) and any \( C_1^- \in C_1 \{1, 4\} \), we have that \( X \in M \{1, 4\} \) if and only if

\[
R(A) \cap R(B) = \{0\}, \quad R(D^*) \subseteq R(B^*).
\]

**Proof.** (a) \( \Rightarrow \) Since \( AC_1^t = 0 \), we have

\[
MX = \begin{pmatrix} AA^- + B_1 B_1^- & 0 \\ (I - C_1 C_1^t)(C A^- + S A B_1^-) & C_1 C_1^t \end{pmatrix},
\]

which implies

\[
(I - C_1 C_1^t)(C A^- + S A B_1^-) = 0.
\]

On the other hand, \( M X M = M \) gives

\[
\begin{pmatrix} A + B_1 B_1^t A & AA^- B + B_1 B_1^- B \\ C_1 C_1^t C & C_1 C_1^t D \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
\]
Hence, $C_1 C_1^\dagger C = C$ and $C_1 C_1^\dagger D = D$ which implies that $R(C) \subseteq R(C_1) \subseteq R(C)$ and $R(D) \subseteq R(C_1) \subseteq R(C)$. Hence, $R(C) = R(C_1)$ i.e. $C_1 C_1^\dagger = CC^\dagger$ which is by Lemma 1.3 equivalent to $R(A^*) \cap R(C^*) = \{0\}$.

$(\Leftarrow)$ By $CC^\dagger D = D$, $C_1 C_1^\dagger = CC^\dagger$, it is easy to verify that $X \in M\{1, 3\}$.

(b) The proof follows by taking transpose conjugate in $(a)$. \hfill \Box

**Theorem 2.7.** Let $M$ be given by (1.2). Let $S_A, B_1, C_1$ be defined by (2.1) and let

$$X = \begin{pmatrix}
A^\dagger - C_1^\dagger C A^\dagger - A^\dagger B B_1^\dagger - C_1^\dagger S_A B_1^\dagger - C_1^\dagger & C_1^\dagger \\
B_1^\dagger & 0
\end{pmatrix}$$

Then the following hold:

(a) If we take that $C_1^\dagger = C_1^\dagger$, then for any $A^\dagger \in A\{1, 2, 3\}$ and any $B_1^\dagger \in B_1\{1, 2, 3\}$ we have that $X \in M\{1, 2, 3\}$ if and only if

$$R(A^*) \cap R(C^*) = \{0\}, \quad R(D) \subseteq R(C).$$

(b) If we take that $B_1^\dagger = B_1^\dagger$, then for any $A^\dagger \in A\{1, 2, 4\}$ and any $C_1^\dagger \in C_1\{1, 2, 4\}$, we have that $X \in M\{1, 2, 4\}$ if and only if

$$R(A) \cap R(B) = \{0\}, \quad R(D^*) \subseteq R(B^*).$$

(c) If we take that $A^\dagger = A^\dagger$, $B_1^\dagger = B_1^\dagger$, $C_1^\dagger = C_1^\dagger$, then $X = M^\dagger$ if and only if

$$R(A^*) \cap R(C^*) = \{0\}, \quad R(A) \cap R(B) = \{0\}, \quad R(D) \subseteq R(C), \quad R(D^*) \subseteq R(B^*).$$

**Proof.** (a) Since $A^\dagger \in A\{1, 2, 3\}$, $B_1^\dagger \in B_1\{1, 2, 3\}$, $X \in M\{1, 2, 3\}$, and hence belong to $A\{1, 3\}$, $B_1\{1, 3\}$, $M\{1, 3\}$, respectively, by Theorem 2.6(a), we get $R(A^*) \cap R(C^*) = \{0\}$, $R(D) \subseteq R(C)$. Conversely, from Theorem 2.6(a), we have $X \in M\{1, 3\}$. It is easy to check $X M X = X$. Thus, $X \in M\{1, 2, 3\}$.

(b) The proof follows from (a) by taking transpose conjugate.

(c) Combining (a) and (b), we get the result. \hfill \Box

Remark that Theorem 2.7 (c) is a generalization of Corollary 2.7 [28]. The next theorem is also a generalization of Corollary 2.8 from [28].

**Theorem 2.8.** Let $M$ be given by (1.2). Then,

$$M^\dagger = \begin{pmatrix}
A^\dagger - C_1^\dagger C A^\dagger - A^\dagger B B_1^\dagger - C_1^\dagger & C_1^\dagger \\
B_1^\dagger & S_A
\end{pmatrix}$$

if and only if

$$R(A^*) \cap R(C^*) = \{0\}, \quad R(A) \cap R(B) = \{0\}, \quad R(S_A^*) \subseteq N(B), \quad R(S_A) \subseteq N(C^*),$$

where $S_A = D - C A^\dagger B$, $B_1 = F_A B$, $C_1 = C E_A$. 

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Proof. (⇒) Since $AC_1^\dagger = 0$, we get

\[ MM^\dagger = \begin{pmatrix}
    AA^\dagger + B_1B_1^\dagger & BS_A^\dagger \\
    (I - C_1C_1^\dagger)CA^\dagger + S_AB_1^\dagger & DS_A^\dagger + C_1C_1^\dagger
\end{pmatrix}. \tag{2.22} \]

By $MM^\dagger M = M$, we get

\[ \begin{pmatrix}
    A + BS_A^\dagger C & AA^\dagger B + B_1B_1^\dagger B + BS_A^\dagger D \\
    (I - C_1C_1^\dagger)CA^\dagger A + DS_A^\dagger C & (I - C_1C_1^\dagger)CA^\dagger B + S_AB_1^\dagger B + DS_A^\dagger D
\end{pmatrix} = \begin{pmatrix}
    A & B \\
    C & D
\end{pmatrix}, \]

Hence

\[ BS_A^\dagger C = 0 \tag{2.23} \]

and

\[ AA^\dagger B + B_1B_1^\dagger B + BS_A^\dagger D = B. \tag{2.24} \]

Since $B_1B_1^\dagger B = B_1B_1^\dagger F_AB = B_1$, by (2.24) we get

\[ BS_A^\dagger D = 0. \]

which together with (2.23) implies that $BS_A^\dagger = 0$, i.e. $R(S_A^\dagger) \subseteq N(B)$. Therefore, (2.22) is simplified to

\[ MM^\dagger = \begin{pmatrix}
    AA^\dagger + B_1B_1^\dagger & 0 \\
    0 & S_AS_A^\dagger + C_1C_1^\dagger
\end{pmatrix}. \tag{2.25} \]

By (2.22) and (2.25), we have

\[ (I - C_1C_1^\dagger)CA^\dagger + S_AB_1^\dagger = 0. \tag{2.26} \]

Multiplying (2.26) by $A$ from the right side we have $(I - C_1C_1^\dagger)CA^\dagger A = 0$, which together with $(I - C_1C_1^\dagger)C_1 = 0$ implies $(I - C_1C_1^\dagger)C = 0$. Hence, $R(C) \subseteq R(C_1) \subseteq R(C)$ i.e. $C_1C_1^\dagger = CC^\dagger$ which is by Lemma 1.3 equivalent to $R(A^\ast) \cap R(C^\ast) = \{0\}$.

On the other hand, using the same method as above, by $M^\dagger M = (M^\dagger M)^\ast$, we get

\[ R(A) \cap R(B) = \{0\}, R(S_A) \subseteq N(C^\ast). \]

(⇐) It follows by Corollary 2.8 [28] \qed

**Theorem 2.9.** Let $M$ be given by (1.2) and let

\[ X = \begin{pmatrix}
    (F_BAE_C)^\dagger & (F_DCCE_A)^\dagger \\
    (F_ABED)^\dagger & (F_CDEB)^\dagger
\end{pmatrix}, \tag{2.27} \]

Then the following are equivalent:

(a) $X \in M\{1\}$;
(b) $X \in M\{1, 2\}$;

(c) $r(M) = r(A) + r(B) + r(C) + r(D)$.

Proof. (a) $\Leftrightarrow$ (b): Obviously.

(a) $\Rightarrow$ (c): A simple computation shows

\[
MX = \begin{pmatrix}
A(F_B A E C) \dagger + B(F_A B E D) \dagger & A(F_D C E A) \dagger + B(F_C D E B) \dagger \\
C(F_B A E C) \dagger + D(F_A B E D) \dagger & C(F_D C E A) \dagger + D(F_C D E B) \dagger
\end{pmatrix}.
\] (2.28)

Obviously, $A(F_D C E A) \dagger = 0$, $B(F_C D E B) \dagger = 0$, $C(F_B A E C) \dagger = 0$, $D(F_A B E D) \dagger = 0$,

so (2.28) is equivalent to

\[
MX = \begin{pmatrix}
A(F_B A E C) \dagger + B(F_A B E D) \dagger & 0 \\
0 & C(F_D C E A) \dagger + D(F_C D E B) \dagger
\end{pmatrix}.
\] (2.29)

By $MXM = M$, we get

\[
\begin{pmatrix}
A(F_B A E C) \dagger A + B(F_A B E D) \dagger & A(F_B A E C) \dagger B + B(F_A B E D) \dagger B \\
C(F_D C E A) \dagger C + D(F_C D E B) \dagger C & C(F_D C E A) \dagger D + D(F_C D E B) \dagger D
\end{pmatrix} = \begin{pmatrix}
A(F_B A E C) \dagger A & B(F_A B E D) \dagger B \\
C(F_D C E A) \dagger C & D(F_C D E B) \dagger D
\end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
\]

Hence, $(F_B A E C) \dagger \in A\{1\}$, $(F_A B E D) \dagger \in B\{1\}$, $(F_D C E A) \dagger \in C\{1\}$, $(F_C D E B) \dagger \in D\{1\}$.

Now, we can say that $A, B, C, D$ are block independent in $\{1\}$-inverse, which by Theorem 4.1 (Pg. 728 [22]) implies that (c) holds.

(c) $\Rightarrow$ (a): By Corollary 2.9 [28], it follows that $X = M^\dagger$.

\[\square\]

3 Application to Sherman-Morrison-Woodbury matrix identity

In the late 1940’s and the 1950’s Sherman and Morrison [30], Woodbury [32], Bartlett [4] and Bodewig [5] discovered the Sherman-Morrison-Woodbury (SMW) formula

\[(A + Y G Z^*)^{-1} = A^{-1} - A^{-1} Y (G^{-1} + Z^* A^{-1} Y)^{-1} Z^* A^{-1}.\]

which was originally used computing ordinary inverses of matrices. Since then the SMW formula has attracted considerable attention and has been widely used in many fields. An excellent paper by Hager [19] describes some of the applications of SMW in statistics, networks, structural analysis, asymptotic analysis, optimization and partial differential equations (see [20, 21]).
In this section, we will extend the SMW formula to a more general case using the results obtained in Section 2. Moreover, we will give some sufficient conditions to present \{1, 2\}, \{1, 3\}, \{1, 4\}, {1, 2, 3}, {1, 2, 4}-inverses by the SMW formula. As corollaries we get some results from [31, 10].

For convenience, we first denote the following notations which are helpful in the following proofs. Let

\[ B_1 = F_{AB}, B_2 = BE_D, C_1 = CE_A, C_2 = F_DC, \]

\[ S_A = D - CA^\dagger B, S_D = A - BD^\dagger C, J_A = F_{2DSE_D}, J_D = F_{E_C}S_AE_D. \]  \quad (3.1)

**Theorem 3.1.** Let \( M \) be given by (1.2). Then,

\[ M^\dagger = \begin{pmatrix} H_1 - H_2CA^\dagger - A^\dagger BH_3 + A^\dagger BJ_D^\dagger & H_2 - A^\dagger BJ_D^\dagger \\ H_3 - J_D^\dagger CA^\dagger & J_D^\dagger \end{pmatrix} \]

if and only if

\[ r(M) = r(A, B) + r(C, D) = r \left( \begin{pmatrix} A \\ C \end{pmatrix} \right) + r \left( \begin{pmatrix} B \\ D \end{pmatrix} \right). \]  \quad (3.2)

where

\[ H_1 = A^\dagger + C_1^\dagger(S_AD^\dagger S_A - S_A)B_1^\dagger, H_2 = C_1^\dagger(I - S_AD^\dagger J_D^\dagger), H_3 = (I - J_D^\dagger S_A)B_1^\dagger, \]

\[ L_1 = D^\dagger + B_2^\dagger(S_DJ_A^\dagger S_D - S_D)C_2^\dagger, L_2 = (I - J_A^\dagger S_D)C_2^\dagger, L_3 = B_2^\dagger(I - S_DJ_A^\dagger). \]

**Proof.** One part of the theorem follows by Theorem 2.2(c). To prove the other part let \( M_1 = \begin{pmatrix} D & C \\ B & A \end{pmatrix} = PMP \), where \( P = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \). Since \( P^2 = I \), we have that \( M_1^\dagger = PM_1^\dagger P \). Now, the other part follows if we apply Theorem 2.2(c) to matrix \( M_1 \). \( \square \)

As a corollary of Theorem 3.1, we get the following result from [28]:

**Corollary 3.1.** [28] If the rank additivity condition (3.2) holds, then

\[ (F_{2DSE_D})^\dagger = A^\dagger + C_1^\dagger(S_AD^\dagger S_A - S_A)B_1^\dagger - C_1^\dagger(I - S_AD^\dagger J_D^\dagger)CA^\dagger \]

\[ -A^\dagger B(I - J_D^\dagger S_A)B_1^\dagger + A^\dagger BJ_D^\dagger CA^\dagger. \]

**Corollary 3.2.** [28] Let \( M \) be given by (1.2), if

\[ R \left( \begin{pmatrix} A \\ 0 \end{pmatrix} \right) \subseteq R(M), R \left( \begin{pmatrix} A^* \\ 0 \end{pmatrix} \right) \subseteq R(M^*), \]

and

\[ R(CSD^\dagger) \subseteq R(D), R(B^*SD) \subseteq R(D^*), \]

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Let \((3.3)\) Suppose that \((3.5)\) holds, then
\[
(A - BD^\dagger C)^\dagger = A^\dagger + C_1^\dagger (S_A J_D^\dagger S_A - S_A) B_1^\dagger - C_1^\dagger (I - S_A J_D^\dagger) C A^\dagger \\
- A^\dagger B (I - J_D^\dagger S_A) B_1^\dagger + A^\dagger B J_D^\dagger C A^\dagger.
\]

**Theorem 3.2.** Let \(M\) be given by (1.2). Then,
\[
M^\dagger = \begin{pmatrix}
A^\dagger - H_2 C A^\dagger & A^\dagger B H_3 + A^\dagger B J_D^\dagger C A^\dagger \\
H_3 - J_D^\dagger C A^\dagger & J_D^\dagger
\end{pmatrix}
\]
\[
= \begin{pmatrix}
J_A^\dagger & L_2 - J_A^\dagger BD^\dagger \\
L_3 - D^\dagger C J_A^\dagger & D^\dagger - D^\dagger C L_2 - L_3 BD^\dagger + D^\dagger C J_A^\dagger BD^\dagger
\end{pmatrix}
\]
if and only if
\[
R \left( \begin{pmatrix} A & 0 \end{pmatrix} \right) \subseteq R(M), R \left( \begin{pmatrix} A^* & 0 \end{pmatrix} \right) \subseteq R(M^*),
\]
\[
R(B_1^\dagger) \cap R(S_A^\dagger) = \{0\}, \quad R(C_1) \cap R(S_A) = \{0\},
\]
\[
R(B_2) \cap R(S_D) = \{0\}, \quad R(C_1^\dagger) \cap R(S_D^\dagger) = \{0\},
\]
where
\[
H_2 = C_1^\dagger (I - S_A J_D^\dagger), H_3 = (I - J_D^\dagger S_A) B_1^\dagger,
\]
\[
L_2 = (I - J_A^\dagger S_D) C_2^\dagger, L_3 = B_2^\dagger (I - S_D J_A^\dagger).
\]

**Proof.** Suppose that (3.4) and (3.5) are satisfied. Since \(r(A, B) = r(A) + r(B) \Leftrightarrow R(A) \cap R(B) = \{0\}\), by (3.5), we get that
\[
r \left( \begin{pmatrix} 0 & B_1 \\ C_1 & S_A \end{pmatrix} \right) = r \left( \begin{pmatrix} B_1 \\ S_A \end{pmatrix} \right) + r(C_1) = r(B_1) + r(C_1, S_A).
\]
Now, by Lemma 1.5 [28] we have that the rank additivity condition (2.6) holds which implies that Theorem 2.3 holds for both matrices \(M\) and \(M_1\). Since (3.5) implies that \(S_A J_D^\dagger S_A = S_A\) and \(S_D J_A^\dagger S_D = S_D\) by Theorem 2.4, \(M^\dagger\) can be represented by (3.3). On the contrary, if \(M^\dagger\) is represented by (3.3), using Theorem 2.4 for matrices \(M\) and \(M_1\), we conclude that (3.4) and (3.5) are satisfied.

As corollaries we get the following results:

**Corollary 3.3.** If the condition (3.4) and (3.5) hold, then
\[
(F B_2 S_D E C_2)^\dagger = A^\dagger - C_1^\dagger (I - S_A J_D^\dagger) C A^\dagger - A^\dagger B (I - J_D^\dagger S_A) B_1^\dagger + A^\dagger B J_D^\dagger C A^\dagger.
\]

**Corollary 3.4.** If \(A, B, C, D\) satisfy the condition (3.4) and
\[
R(B_1^\dagger) \cap R(S_A^\dagger) = \{0\}, R(C_1) \cap R(S_A) = \{0\},
\]
\[ R(CS_D^2) \subseteq R(D), R(B^*S_D) \subseteq R(D^*), \]

hold, then

\[ \begin{align*}
(A - BD^1C)^\dagger &= A^\dagger - C_1^\dagger(I - S_AJ_D^1)CA^\dagger - A^\dagger B(I - J_D^1S_A)B_1^\dagger + A^\dagger BJ_D^1CA^\dagger. 
\end{align*} \]  \tag{3.6} \]

**Proof.** From \( R(CS_D^2) \subseteq R(D), R(B^*S_D) \subseteq R(D^*) \) we get that

\[ S_D(F_D C)^\dagger = S_D C_2^\dagger = 0, \quad (BE_D)^\dagger S_D = B_2^\dagger S_D = 0 \]  \tag{3.7} \]

which implies that

\[ R(B_2) \cap R(S_D) = \{0\}, R(C_1^\dagger) \cap R(S_D^2) = \{0\}. \]

Thus, the condition (3.4) and (3.5) are satisfied. Using (3.7), we get that \( F_{B_2}S_DE_{C_2} = A - BD^1C^\dagger \). Thus, we get (3.6) by Corollary 3.3.

According to Theorem 2.4, we can get the following results.

**Theorem 3.3.** Let \( M \) be given by (1.2). Then,

\[ M^\dagger = \begin{pmatrix}
A^\dagger - C_1^\dagger CA^\dagger - A^\dagger BB_1^\dagger + A^\dagger BS_A^1CA^\dagger & C_1^\dagger - A^\dagger BS_A^1 \\
B_1^\dagger - S_A^1CA^\dagger & S_A^1 \\
C_2^\dagger - S_D^1BD^1 & S_D^1 \\
B_2^\dagger - D^1CS_D^1 & D^\dagger - D^1CC_2^\dagger - B_2^\dagger BD^\dagger + D^1CS_D^1BD^\dagger
\end{pmatrix} \]

if and only if

\[ R \begin{pmatrix} A \\ 0 \end{pmatrix} \subseteq R(M), R \begin{pmatrix} A^* \\ 0 \end{pmatrix} \subseteq R(M^*), \]

\[ R(BS_A^*) \subseteq R(A), R(C^*S_A) \subseteq R(A^*), R(CS_D^2) \subseteq R(D), R(B^*S_D) \subseteq R(D^*). \]  \tag{3.8} \tag{3.9} \]

**Corollary 3.5.** If the condition (3.8) and (3.9) holds, then

\[ (A - BD^1C)^\dagger = A^\dagger - C_1^\dagger CA^\dagger - A^\dagger BB_1^\dagger + A^\dagger BS_A^1CA^\dagger. \]

In the following, we establish the sufficient conditions for the Sherman-Morrison-Woodbury Formula formula of \( \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}\)-inverses. First, we denote

\[ E_\alpha = I - \alpha^\dagger, F_\alpha = I - \alpha_\dagger, S = D - CA_\dagger B, G = A - BD_\dagger C, \]

where \( \alpha^\dagger \in A\{1\}, A_\dagger \in A\{1\}, D_\dagger \in D\{1\}. \)

**Theorem 3.4.** For any \( A^- \in A\{1, 3\}, S^- \in S\{1\} \) and \( D^- \in D\{1\} \) such that \( F_A B = 0, F_S C = 0, BE_D = 0, \) we have

\[ A^- + A^- B(D - CA^- B)^\dagger CA^- \in G\{1, 3\}. \]
Proof. Let \( X = A^- + A^- B(D - C A^- B)^- C A^- \). Since \( F_A B = 0, \ F_S C = 0, \ B E_D = 0 \), a simple computation shows that

\[
GX = (A - BD^- C)(A^- + A^- B(D - C A^- B)^- C A^-) \\
= AA^- + AA^- BS^- C A^- - BD^- CA^- - BD^- CA^- BS^- C A^- \\
= AA^- + AA^- BS^- C A^- - BD^- CA^- - BD^- (D - S) S^- C A^- \\
= AA^- + AA^- BS^- C A^- - BD^- CA^- - BD^- DS^- C A^- + BD^- SS^- C A^- \\
= AA^-.
\]

Thus, \( GX \) is Hermitian. On the other hand, by \( F_A B = 0 \), we get

\[
GXG = AA^- (A - BD^- C) = A - BD^- C = G.
\]

Therefore, \( X \in G\{1, 3\} \). \( \square \)

**Theorem 3.5.** For any \( A^- \in A\{1, 4\}, \ S^- \in S\{1\} \) and \( D^- \in D\{1\} \) such that \( C E_A = 0, \ F_D C = 0, \ B E_S = 0 \), we have

\[
A^- + A^- B(D - C A^- B)^- C A^- \in G\{1, 4\}.
\]

**Proof.** Let \( Y = A^- + A^- B(D - C A^- B)^- C A^- \). Then

\[
YG = (A^- + A^- B(D - C A^- B)^- C A^-)(A - BD^- C) \\
= A^- A.
\]

Thus, \( YG \) is Hermitian. Moreover, by \( C E_A = 0 \), we get

\[
GYG = (A - BD^- C) A^- A = A - BD^- C = G.
\]

Therefore, \( Y \in G\{1, 4\} \). \( \square \)

**Theorem 3.6.** If one of the following conditions

(a) \( F_A B = 0, \ F_S C = 0, \ B E_D = 0 \);

(b) \( C E_A = 0, \ F_D C = 0, \ B E_S = 0 \)

holds, for some \( A^- \in A\{1, 2\}, \ S^- \in S\{1\} \) and \( D^- \in D\{1\} \), then

\[
A^- + A^- B(D - C A^- B)^- C A^- \in G\{1, 2\}. \tag{3.10}
\]

**Proof.** According to the proofs of Theorem 3.4 and Theorem 3.5, it is easy to verify that (3.10) holds. \( \square \)

By the results above, we get the result in the case of \( \{1, 2, 3\} \) and \( \{1, 2, 4\} \)-inverses.
Theorem 3.7. For any $A^- \in A\{1,2,3\}$, $S^- \in S\{1\}$ and $D^- \in D\{1\}$ such that $F_A B = 0$, $F_S C = 0$, $BE_D = 0$, we have that
\[ A^- + A^- B (D - CA^- B)^- C A^- \in G\{1,2,3\}. \]

Theorem 3.8. For any $A^- \in A\{1,2,4\}$, $S^- \in S\{1\}$ and $D^- \in D\{1\}$ such that $C E_A = 0$, $F_D C = 0$, $BE_S = 0$, we have that
\[ A^- + A^- B (D - CA^- B)^- C A^- \in G\{1,2,4\}. \]

References


[6] Y. Chen and B. Zhou, On $g$-inverses and nonsingularity of a bordered matrix $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$, Linear Algebra Appl. 133 (1990), 133-151.


