New results on reverse order law for $\{1, 2, 3\}$ and $\{1, 2, 4\}$-inverses of bounded operators

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Abstract

In this paper, using some block-operator matrix techniques, we give the necessary and sufficient conditions for the reverse order law for $\{1, 2, 3\}$ and $\{1, 2, 4\}$-inverses of bounded operators on Hilbert spaces. Furthermore, we present new equivalent conditions for the reverse order law for the Moore-Penrose inverse.

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1 Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces and let $\mathcal{B}(\mathcal{H}, \mathcal{K})$ denote the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. For a given $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, the symbols $N(A)$ and $\mathcal{R}(A)$ denote the null space and the range of $A$, respectively. For a given sets $M$, $N$, by $MN$ we denote the set consisting of all products $XY$, where $X \in M$ and $Y \in N$.

Recall that $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ has a Moore-Penrose inverse if there exists an operator $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that

\begin{align}
(1) \quad AXA &= A \\
(2) \quad XAX &= X \\
(3) \quad (AX)^* &= AX \\
(4) \quad (XA)^* &= XA.
\end{align} \tag{1.1}

Moore-Penrose inverse of an operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ exists if and only if $A$ has a closed range and in this case it is unique. It is denoted by $A^\dagger$.

For any $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, let $A\{i, j, \ldots, k\}$ denote the set of operators $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ which satisfies equations $(i)$, $(j)$, ..., $(k)$ from among equations $(1)$–$(4)$ of (1.1). In this case $X$ is $\{i, j, \ldots, k\}$–inverse of $A$ which is denoted by $A^{(i,j,\ldots,k)}$. Evidently, $A\{1, 2, 3, 4\} = \{A^\dagger\}$, when $A$ has a closed range.

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The reverse order laws for two matrices or operators have been investigated intensively (see [5], [6], [9], [10], [12]-[14]). T.N.E. Greville [8] proved that $(AB)^\dagger = B^\dagger A^\dagger$ if and only if $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$, for matrices $A$ and $B$. This result was extended to linear bounded operators on Hilbert spaces in [10]. Later, the reverse order law for the Moore-Penrose inverse was considered in rings with involution (see [11]).

Xiong and Zheng [17] considered the reverse order law for $\{1, 2, 3\}$ and $\{1, 2, 4\}$—generalized inverses of the products of two matrices and their techniques involved expressions for maximal and minimal ranks of the generalized Schur complement. In [2] authors considered the reverse order law for $K$-inverses in the cases $K \in \{\{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}\}$ for the elements of C*-algebra.

In this paper, using block-operator matrix techniques, we consider the reverse order law for $\{1, 2, 3\}$ and $\{1, 2, 4\}$—inverses of bounded operators on Hilbert spaces. We give the necessary and sufficient conditions for

$$B\{1, 2, 3\} \cdot A\{1, 2, 3\} \subseteq (AB)\{1, 2, 3\}$$

and

$$B\{1, 2, 4\} \cdot A\{1, 2, 4\} \subseteq (AB)\{1, 2, 4\}.$$ We generalized the results from [2] for the case of bounded operators on Hilbert space. Furthermore, we present new equivalent conditions for the reverse order law for the Moore-Penrose inverse.

## 2 Preliminaries

Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ has a closed range. The operator $A$ has the following matrix decomposition (see [4], [7])

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}, \tag{2.1}$$

where $A_1$ is invertible. Also $A^\dagger$ has the form

$$A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}. \tag{2.2}$$

If $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ has a closed range, then we can explicitly describe the sets $A\{1, 2, 3\}$ and $A\{1, 2, 4\}$ using the representation of $A$ given by (2.1).

**Lemma 2.1.** Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ has a closed range. Then

$$A\{1, 2, 3\} = \left\{ \begin{bmatrix} A_1^{-1} & 0 \\ X_3 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} : X_3 \in \mathcal{B}(\mathcal{R}(A), \mathcal{N}(A)) \right\}$$

and

$$A\{1, 2, 4\} = \left\{ \begin{bmatrix} A_1^{-1} & X_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} : X_2 \in \mathcal{B}(\mathcal{N}(A^*), \mathcal{R}(A^*)) \right\}.$$
Proposition. Suppose that $A$ and $A^\dagger$ are given by (2.1) and (2.2), respectively. Since
\[
A\{1, 2, 3\} = \{A^\dagger + (I - A^\dagger A)XAA^\dagger : X \in \mathcal{B}(\mathcal{K}, \mathcal{H})\},
\]
we have that $A^{(1,2,3)} \in A\{1, 2, 3\}$ if and only if
\[
A^{(1,2,3)} = A^\dagger + (I - A^\dagger A)XAA^\dagger
= \begin{bmatrix}
A_1^{-1} & 0 \\
0 & 0 \\
X_3 & X_4
\end{bmatrix}
\begin{bmatrix}
X_1 & X_2 \\
X_3 & X_4
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
0 & 0
\end{bmatrix}
= \begin{bmatrix}
A_1^{-1} & 0 \\
X_3 & 0
\end{bmatrix},
\]
for some $X = \begin{bmatrix}
X_1 & X_2 \\
X_3 & X_4
\end{bmatrix} : \begin{bmatrix}
\mathcal{R}(A) \\
\mathcal{N}(A)
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{R}(A^*) \\
\mathcal{N}(A)
\end{bmatrix}$. The proof for the case of $\{1, 2, 4\}$-inverses follows analogous.

\[\square\]

Lemma 2.2. Let $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{H}$ be Hilbert spaces. Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ be such that $\mathcal{R}(A)$, $\mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed. Then $\mathcal{R}(B^*) \cap \mathcal{N}(AB) = \{0\}$ if and only if $\mathcal{R}(B) \cap \mathcal{N}(A) = \{0\}$.

Proof. First, let us remark that $B|_{\mathcal{R}(B^*)} : \mathcal{R}(B^*) \rightarrow \mathcal{R}(B)$ is invertible operator.

\[\begin{aligned}
(\Rightarrow) : \text{Suppose that } & \mathcal{R}(B^*) \cap \mathcal{N}(AB) = \{0\} \text{ and let } x \in \mathcal{R}(B) \cap \mathcal{N}(A). \text{ By (2.3), there exists } y \in \mathcal{R}(B^*) \text{ such that } By = x. \text{ Now, } y \in \mathcal{R}(B^*) \cap \mathcal{N}(AB) \text{ i.e. } y = 0, \text{ so } x = By = 0.

(\Leftarrow) : \text{If we suppose that } & \mathcal{R}(B) \cap \mathcal{N}(A) = \{0\} \text{ and take } u \in \mathcal{R}(B^*) \cap \mathcal{N}(AB), \text{ we get that } Bu \in \mathcal{R}(B) \cap \mathcal{N}(A) \text{ i.e. } Bu = 0. \text{ Using (2.3) it follows that } u = 0. \square
\end{aligned}\]

Let us introduce the following notations: If a Hilbert space $\mathcal{H}$ is decomposed as $\mathcal{H} = U_1 \oplus \cdots \oplus U_k$ where $U_i \perp U_j$ for $i \neq j$, then we shall denote that $\mathcal{H} = U_1 \oplus^\perp \cdots \oplus^\perp U_k$. If $\mathcal{U}$ is a complement space of a Hilbert space $\mathcal{H}$ we shall denote by $\mathcal{H} \ominus^\perp \mathcal{U}$ the unique subspace $\mathcal{V}$ of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{U} \oplus^\perp \mathcal{V}$.

Remark 2.1. Let $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{H}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{K}, \mathcal{K})$, $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ be such that $\mathcal{R}(A)$, $\mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed. Denote by
\[
\begin{align*}
\mathcal{H}_1 &= \mathcal{R}(B) \cap \mathcal{N}(A), \\
\mathcal{H}_2 &= \mathcal{R}(B) \ominus^\perp \mathcal{H}_1, \\
\mathcal{H}_3 &= \mathcal{N}(B^*) \cap \mathcal{N}(A), \\
\mathcal{H}_4 &= \mathcal{N}(B^*) \ominus^\perp \mathcal{H}_3,
\end{align*}
\]
Hilbert spaces $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{H}$ can be decomposed as
\[
\mathcal{K} = \mathcal{R}(B) \ominus^\perp \mathcal{N}(B^*), \quad \mathcal{K} = \mathcal{R}(A) \ominus^\perp \mathcal{N}(A^*), \quad \mathcal{L} = \mathcal{R}(B^*) \ominus^\perp \mathcal{N}(B),
\]
where
\[
\mathcal{R}(B) = \mathcal{H}_1 \ominus \mathcal{H}_2, \quad \mathcal{N}(B^*) = \mathcal{H}_3 \ominus^\perp \mathcal{H}_4, \quad \mathcal{R}(A) = \mathcal{K}_1 \ominus \mathcal{K}_2, \quad \mathcal{R}(B^*) = \mathcal{L}_1 \ominus^\perp \mathcal{L}_2.
\]

3
We can prove that \( B^\dagger (\mathcal{R}(B) \cap N(A)) = \mathcal{R}(B^*) \cap N(AB) \). Let \( x \in \mathcal{R}(B^*) \cap N(AB) \). Then \( x \in \mathcal{R}(B^\dagger) = \mathcal{R}(B^\dagger B) = B^\dagger \mathcal{R}(B) \) and \( ABx = 0 \), i.e., \( Bx \in N(A) \). So we have \( x = B^\dagger Bx \in B^\dagger N(A) \). Finally, \( x \in B^\dagger (\mathcal{R}(B) \cap N(A)) \). On the other hand, let \( y \in B^\dagger (\mathcal{R}(B) \cap N(A) \), i.e., \( y \in \mathcal{R}(B^\dagger B) = \mathcal{R}(B^*) \) and there exists \( z \in \mathcal{R}(B) \cap N(A) \) such that \( y = B^\dagger z \), then \( ABy = ABB^\dagger z = Az = 0 \), i.e., \( y \in N(AB) \). Thus, by Lemma 2.2, we get

\[
\mathcal{H}_2 = \mathcal{R}(B) \leftrightarrow \mathcal{H}_1 = \{0\} \leftrightarrow N(AB) = N(B) \leftrightarrow \mathcal{L}_1 = \{0\} \leftrightarrow \mathcal{L}_2 = \mathcal{R}(B^*). 
\]

Furthermore,

\[
\mathcal{H}_2 = \{0\} \leftrightarrow \mathcal{H}_1 = \mathcal{R}(B) \leftrightarrow \mathcal{R}(B) \subset N(A) \leftrightarrow \mathcal{L}_1 = \mathcal{R}(B^*) \leftrightarrow \mathcal{L}_2 = \{0\} \leftrightarrow \mathcal{K}_1 = \{0\} \leftrightarrow \mathcal{K}_2 = \mathcal{R}(A).
\]

Throughout the paper we will use the notations from the above remark.

A similar result to the following one but for the case of two Hilbert spaces has been presented in [14]. Now we give a different proof in the case of three Hilbert spaces.

**Lemma 2.3.** Let \( \mathcal{H}, \mathcal{K} \) and \( \mathcal{L} \) be Hilbert spaces and let \( A \in \mathcal{B}(\mathcal{H}, \mathcal{K}), B \in \mathcal{B}(\mathcal{L}, \mathcal{H}) \) be such that \( \mathcal{R}(A), \mathcal{R}(B) \) and \( \mathcal{R}(AB) \) are closed.

(1) If \( AB \neq 0 \) and \( N(AB) \neq N(B) \), then \( A \) and \( B \) have the following operator matrix forms

\[
A = \begin{bmatrix}
0 & A_{12} & 0 & A_{14} \\
0 & 0 & 0 & A_{24} \\
0 & 0 & 0 & 0
\end{bmatrix} : \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{K}_3 \\
N(A^*)
\end{bmatrix}, \tag{2.4}
\]

and

\[
B = \begin{bmatrix}
B_{11} & B_{12} & 0 \\
0 & B_{22} & 0 \\
0 & 0 & 0
\end{bmatrix} : \begin{bmatrix}
\mathcal{L}_1 \\
\mathcal{L}_2 \\
N(B)
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix}, \tag{2.5}
\]

where \( A_{12}, B_{11}, B_{22} \) are invertible operators and \( A_{24} \) is a surjection.

(2) If \( AB \neq 0 \) and \( N(AB) = N(B) \), then \( A \) and \( B \) have the following operator matrix forms

\[
A = \begin{bmatrix}
A_{12} & 0 & A_{14} \\
0 & 0 & A_{21} \\
0 & 0 & 0
\end{bmatrix} : \begin{bmatrix}
\mathcal{R}(B) \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
N(A^*)
\end{bmatrix}, \tag{2.6}
\]

and

\[
B = \begin{bmatrix}
B_{22} & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} : \begin{bmatrix}
\mathcal{R}(B^*) \\
N(B)
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{R}(B) \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix}, \tag{2.7}
\]

where \( A_{12}, B_{22} \) are invertible operators and \( A_{24} \) is a surjection.
(3) If $AB = 0$ and $N(AB) \neq N(B)$, then $A$ and $B$ have the following operator matrix forms

$$A = \begin{bmatrix} 0 & 0 & A_{24} \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{H}(B) \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ N(A^*) \end{bmatrix}$$

and

$$B = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{21} & B_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ N(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},$$

where $B_{11}$ is invertible operator and $A_{24}$ is invertible.

Proof. We will assume that the spaces $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ are decomposed as in the Remark 2.1, so the conclusions from that remark also hold.

(1) Suppose that $AB \neq 0$ and $N(AB) \neq N(B)$. We have that $B$ can be represented by

$$B = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{21} & B_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ N(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},$$

where $\widehat{B} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix}$ is invertible.

Since $B\mathcal{L}_1 \subset \mathcal{H}_1$, we get that $B_{21} = 0$. Now from the invertibility of $\widehat{B}$, we get that $B_{11} : \mathcal{L}_1 \to \mathcal{H}_1$ and $B_{22} : \mathcal{L}_2 \to \mathcal{H}_2$ are invertible.

Now, we will prove that $A$ has a matrix form given by (2.4). Suppose that

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \to \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \\ N(A^*) \end{bmatrix}.$$

The reductions $A_{i3}$ for $i = 1, 2, 3$ are null operators because $\mathcal{H}_1, \mathcal{H}_3 \subset N(A)$. The range of the reductions $A_{2j}$, $(j = 1, 2, 3, 4)$ is $N(A^*)$, so $A_{3j} = 0$.

Now we will prove that $A_{22} = 0$: for any $x \in \mathcal{H}_2 \subset \mathcal{R}(B)$, there exists $y \in \mathcal{K}$ such that $By = x$. Now, $Ax = ABy \in \mathcal{K}_1$ and $Ax = A_{12}x + A_{22}x$. Since $A_{12}x \in \mathcal{K}_1$, we get that $A_{22}x = 0$.

In order to prove that $A_{12}$ is bijective, first we will prove that $N(A_{12}) = \{0\}$: let $u \in \mathcal{H}_2$ be such that $A_{12}u = 0$. Then $u \in N(A)$ which implies that $u \in \mathcal{K}_1 \cap \mathcal{H}_2 = \{0\}$.

To prove that $A_{12} : \mathcal{H}_2 \to \mathcal{K}_1$ is surjective take any $k \in \mathcal{K}_1 = \mathcal{R}(AB)$. There exists $k' \in \mathcal{K}$ such that $ABk' = k$. Since $Bk' \in \mathcal{R}(B) = \mathcal{H}_1 \oplus \mathcal{H}_2$, there exist $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$ such that $Bk' = h_1 + h_2$. Now, $Ah_2 = A(Bk' - h_1) = k$ i.e. $A_{12}h_2 = k$.

The surjective properties of $A_{24} : \mathcal{H}_4 \to \mathcal{K}_2$ follows from the fact that for any $u \in \mathcal{K}_2$, there exists $v \in \mathcal{H}$ such that $Av = u$. Let us decompose $v = \sum_{i=1}^{4} v_i$, where $v_i \in \mathcal{H}_i$. It is evident that $A_{24}v_4 = u$.

The proof of (2) and (3) is analogous.
3 Main results

Z. Xiong and B. Zheng [17] presented necessary and sufficient conditions for
\[ B\{1, 2, 3\}A\{1, 2, 3\} \subseteq (AB)\{1, 2, 3\}, \tag{3.1} \]
in the case when \( A \) and \( B \) are matrices. Here, we give another characterization of (3.1) for linear bounded operators on Hilbert spaces using techniques which are completely different from those used in [17]. First, we will give the following remark:

**Remark 3.1.** Let \( \mathcal{H} \), \( \mathcal{K} \) and \( \mathcal{L} \) be Hilbert spaces and let \( A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \), \( B \in \mathcal{B}(\mathcal{L}, \mathcal{H}) \) be such that \( \mathcal{R}(A) \), \( \mathcal{R}(B) \) and \( \mathcal{R}(AB) \) are closed, \( AB \neq 0 \) and \( N(AB) \neq N(B) \). Then we can suppose that the operators \( A \) and \( B \) are represented by (2.4) and (2.5), respectively. By Lemma 2.1, \( X \in B\{1, 2, 3\} \) if and only if there exist operators \( F_{11} \) and \( F_{12} \) such that

\[
X = \begin{bmatrix}
B_{11}^{-1} - B_{12}^{-1}B_{12}B_{22}^{-1} & 0 & 0 \\
0 & B_{22}^{-1} & 0 \\
F_{11} & F_{12} & 0 
\end{bmatrix}
: \begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{K}_3 \\
\mathcal{K}_4 
\end{bmatrix} \to \begin{bmatrix}
\mathcal{L}_1 \\
\mathcal{L}_2 \\
\mathcal{N}(B) 
\end{bmatrix}. \tag{3.2} \]

To describe the set \( A\{1, 2, 3\} \), suppose that an arbitrary \( Y \in A\{1, 2, 3\} \) is given by

\[
Y = \begin{bmatrix}
Y_{11} & Y_{12} & Y_{13} \\
Y_{21} & Y_{22} & Y_{23} \\
Y_{31} & Y_{32} & Y_{33} \\
Y_{41} & Y_{42} & Y_{43} 
\end{bmatrix}
: \begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{K}_3 \\
\mathcal{K}_4 
\end{bmatrix} \to \begin{bmatrix}
\mathcal{N}(A^*) 
\end{bmatrix}. \tag{3.3} \]

Since \( AY \) is hermitian, we get that

\[
AY = \begin{bmatrix}
A_{12}Y_{21} + A_{14}Y_{41} & A_{12}Y_{22} + A_{14}A_{42} \\
A_{24}Y_{41} & A_{24}Y_{42} 
\end{bmatrix}
: \begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{K}_3 \\
\mathcal{K}_4 
\end{bmatrix} \to \begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{N}(A^*) 
\end{bmatrix}
\]

where \( A_{12}Y_{22} + A_{14}A_{42} = (A_{24}Y_{41})^* \) and \( A_{12}Y_{21} + A_{14}Y_{41} \), \( A_{24}Y_{42} \) are hermitian. Since \( AY \) is orthogonal projection on \( \mathcal{R}(A) \), from the definition of the subspaces \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) we can conclude that \( A_{12}Y_{21} + A_{14}Y_{41} = I, A_{24}Y_{42} = I, A_{12}Y_{22} + A_{14}A_{42} = 0 \) and \( A_{24}Y_{42} = 0 \). Now, from \( YAY = 0 \), we get that \( Y_{i3} = 0 \), for \( i = 1, 4 \). Hence, \( Y \in A\{1, 2, 3\} \) if and only if

\[
Y = \begin{bmatrix}
Y_{11} & Y_{12} & 0 \\
Y_{21} & Y_{22} & 0 \\
Y_{31} & Y_{32} & 0 \\
Y_{41} & Y_{42} & 0 
\end{bmatrix}
: \begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{K}_3 \\
\mathcal{K}_4 
\end{bmatrix} \to \begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{N}(A^*) 
\end{bmatrix}, \tag{3.4} \]

where \( Y_{ij} \) satisfy the following equalities

\[
\begin{aligned}
Y_{i2}A_{24}Y_{42} &= Y_{i2}, \quad i = 1, 4, \\
A_{12}Y_{21} + A_{14}Y_{41} &= I_{\mathcal{K}_1}, \\
A_{12}Y_{22} + A_{14}A_{42} &= 0, \\
A_{24}Y_{42} &= I_{\mathcal{K}_2}, A_{24}Y_{41} = 0.
\end{aligned} \tag{3.5} \]
Let $B$, $R$. We use the decompositions of the spaces $Y$ first equation of (3.5), we get that $Y$ and $N$ respectively. Then $Y$ are uniquely determined. Since $A$, $B$, $Y$ and $N$ are closed and $M$, $A$, $B$, $Y$ and $N$ must be of the form $A_{i1}$, $A_{12}$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21$, $22$, $12$, $21
(ii) ⇒ (i): Suppose $\mathcal{R}(A^*AB) = \mathcal{R}(B) \oplus \text{Im} [\mathcal{R}(B) \cap \mathcal{N}(A)]$ and $\mathcal{R}(AB) = \mathcal{R}(A)$. We must show that for arbitrary $X \in B\{1,2,3\}$ and $Y \in A\{1,2,3\}$ there exists $Z \in (AB)\{1,2,3\}$ such that $XY = Z$.

From $\mathcal{R}(AB) = \mathcal{R}(A)$, we get that $\mathcal{K}_2 = \{0\}$, i.e. $A_{24} = 0$. Also by $\mathcal{R}(A^*AB) = \mathcal{R}(B) \oplus \text{Im} [\mathcal{R}(B) \cap \mathcal{N}(A)]$ and the fact that

$$A^*AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{12}^*A_{12}B_{22} & 0 \\ 0 & 0 & 0 \\ 0 & A_{14}^*A_{12}B_{22} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix}$$

where $A_{12}$ and $B_{22}$ are invertible we have $A_{14} = 0$.

Now, we get that $Y \in A\{1,2,3\}$ if and only if

$$Y = \begin{bmatrix} Y_{11} & 0 & 0 \\ A_{12}^{-1} & 0 & 0 \\ Y_{31} & 0 & 0 \\ Y_{41} & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix}, \quad (3.9)$$

where $Y_{11}, Y_{31}, Y_{41}$ are arbitrary. It is evident that for arbitrary $X \in B\{1,2,3\}$ and $Y \in A\{1,2,3\}$ there exists $Z \in (AB)\{1,2,3\}$ such that $XY = Z$, i.e. $B\{1,2,3\}A\{1,2,3\} \subseteq (AB)\{1,2,3\}$.

(2) When $\mathcal{N}(AB) = \mathcal{N}(B)$, the operators $A$ and $B$ are represented by (2.6) and (2.7), respectively and the proof is analogous to the case (1).

\[ \square \]

**Remark 3.2.** 1° If $AB = 0$, then $(AB)\{1,2,3\} = \{0\}$. In the case when $A = 0$ or $B = 0$, evidently $B\{1,2,3\}A\{1,2,3\} \subseteq (AB)\{1,2,3\}$. If it is not the case, we have that $AB = 0 \leftrightarrow \mathcal{K}_2 = \{0\} \leftrightarrow \mathcal{H}_1 = \mathcal{R}(B) \leftrightarrow \mathcal{L}_2 = \{0\} \leftrightarrow \mathcal{L}_4 = \mathcal{R}(B^*) \leftrightarrow \mathcal{K}_1 = \{0\} \leftrightarrow \mathcal{X}_2 = \mathcal{R}(A)$. Also, $A$ and $B$ are represented by (2.8) and (2.9), respectively, so arbitrary $X \in B\{1,2,3\}$ and $Y \in A\{1,2,3\}$ are represented by

$$X = \begin{bmatrix} B_{11}^{-1} & 0 & 0 \\ F_1 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{N}(B) \end{bmatrix},$$

and

$$Y = \begin{bmatrix} F_2 & 0 \\ F_3 & 0 \\ A_{24}^{-1} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},$$

for some operators $F_1, F_2$ and $F_3$.

By a simple computation, we observe that

$$XY = \begin{bmatrix} B_{11}^{-1}F_2 & 0 \\ F_1F_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{N}(B) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \neq 0,$$

i.e., $B\{1,2,3\}A\{1,2,3\} \neq \{0\}$.

Hence, $AB = 0, A \neq 0, B \neq 0 \Rightarrow B\{1,2,3\}A\{1,2,3\} \not\subseteq (AB)\{1,2,3\}$.
2° From Theorem 3.1 we conclude that the condition
\[(ABB^\dagger)AABB^\dagger = BB^\dagger \quad \text{or} \quad (AB)(AB)^\dagger = AA^\dagger\]
from Theorem 3.3 [2] can be replaced by the sole condition \((AB)(AB)^\dagger = AA^\dagger\) i.e. \(\mathcal{R}(AB) = \mathcal{R}(A)\).

A similar result in the case \(K = \{1, 2, 4\}\) follows from Theorem 3.1 by the reversal of the products:

**Theorem 3.2.** Let \(\mathcal{H}, \mathcal{K}\) and \(\mathcal{L}\) be Hilbert spaces and let \(A \in \mathcal{B}(\mathcal{H}, \mathcal{K}), B \in \mathcal{B}(\mathcal{L}, \mathcal{H})\) be such that \(\mathcal{R}(A), \mathcal{R}(B)\) and \(\mathcal{R}(AB)\) are closed and \(AB \neq 0\). Then the following statements are equivalent:

(i) \(B\{1, 2, 4\}A\{1, 2, 4\} \subseteq (AB)\{1, 2, 4\}\).

(ii) \(\mathcal{R}(A^*) = \mathcal{R}(BB^*A^*) \oplus \mathcal{N}(B^*)\), \(\mathcal{N}(AB) = \mathcal{N}(B)\)

**Remark 3.3.** Let \(\mathcal{H}, \mathcal{K}\) and \(\mathcal{L}\) be Hilbert spaces and let \(A \in \mathcal{B}(\mathcal{H}, \mathcal{K}), B \in \mathcal{B}(\mathcal{L}, \mathcal{H})\) be such that \(\mathcal{R}(A), \mathcal{R}(B), \mathcal{R}(AB)\) are closed, \(AB \neq 0\) and \(\mathcal{N}(AB) \neq \mathcal{N}(B)\). We have that operator \(B\) is represented by (2.5), so

\[
B^\dagger = \begin{bmatrix}
B_{11}^{-1} & -B_{11}^{-1}B_{12}B_{22}^{-1} & 0 & 0 \\
0 & B_{22}^{-1} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} : 
\begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{K}_3 \\
\mathcal{K}_4 \\
\end{bmatrix} \rightarrow 
\begin{bmatrix}
\mathcal{L}_1 \\
\mathcal{L}_2 \\
\mathcal{N}(B) \\
\end{bmatrix}.
\] (3.10)

Also, if we suppose that the operator \(A\) is represented by (2.5), using the representation

\[
A^\dagger = 
\begin{bmatrix}
0 & Y_{12} & 0 \\
Y_{21} & Y_{22} & 0 \\
0 & Y_{32} & 0 \\
Y_{41} & Y_{42} & 0 \\
\end{bmatrix} : 
\begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{N}(A^*) \\
\end{bmatrix} \rightarrow 
\begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{K}_3 \\
\mathcal{K}_4 \\
\end{bmatrix}.
\] (3.11)

where

\[
\begin{aligned}
A_{12}Y_{21} + A_{14}Y_{41} &= I, \\
A_{12}Y_{22} + A_{14}Y_{42} &= 0, \\
A_{23}Y_{32} &= I, A_{23}Y_{41} = 0, \\
Y_{41}A_{14} + Y_{22}A_{24} &= I, \\
Y_{11}A_{12} &= 0, Y_{31}A_{12} = 0, \\
(Y_{21}A_{14} + Y_{22}A_{24})^* &= Y_{41}A_{12}, \\
Y_{11}A_{14} + Y_{12}A_{24} &= 0, \\
Y_{31}A_{14} + Y_{32}A_{24} &= 0, \\
Y_{21}A_{12} &= I.
\end{aligned}
\] (3.12)

A simple computation shows that

\[
B^\dagger A^\dagger = 
\begin{bmatrix}
-B_{11}^{-1}B_{12}B_{22}^{-1}Y_{21} & M_3 & 0 \\
B_{22}^{-1}Y_{21} & B_{22}^{-1}Y_{22} & 0 \\
0 & 0 & 0 \\
\end{bmatrix} : 
\begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{N}(A^*) \\
\end{bmatrix} \rightarrow 
\begin{bmatrix}
\mathcal{L}_1 \\
\mathcal{L}_2 \\
\mathcal{N}(B) \\
\end{bmatrix},
\] (3.13)

where \(M_3 = B_{11}^{-1}Y_{12} - B_{11}^{-1}B_{12}B_{22}^{-1}Y_{22}\).
Using the previous remark, we obtain the following result:

**Theorem 3.3.** Let \( \mathcal{H}, \mathcal{K} \) and \( \mathcal{L} \) be Hilbert spaces and let \( A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \), \( B \in \mathcal{B}(\mathcal{L}, \mathcal{K}) \) be such that \( \mathcal{R}(A), \mathcal{R}(B), \mathcal{R}(AB) \) are closed and \( AB \neq 0 \). Then the following statements are equivalent:

(i) \( B^\dagger A^\dagger \in (AB)\{1, 2, 3\} \).

(ii) \( B\{1, 2, 3\}A^\dagger \subseteq (AB)\{1, 2, 3\} \).

(iii) \( \mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus [\mathcal{R}(B) \cap \mathcal{N}(A)] \).

**Proof.** As in Theorem 3.1, we will use the decompositions of spaces \( \mathcal{H}, \mathcal{K} \) and \( \mathcal{L} \). Also, we distinguish two cases:

(1) Let \( N(AB) \neq N(B) \).

(i)\(\Rightarrow\)(iii) If \( B^\dagger A^\dagger \in (AB)\{1, 2, 3\} \), then there exists an operator \( Z \in (AB)\{1, 2, 3\} \) such that \( B^\dagger A^\dagger = Z \), where \( Z \) is represented by (3.7). Comparing (3.7) with (3.13), we obtain \( Y_{21} = A_{12}^{-1}, Y_{22} = 0, Y_{12} = 0 \).

We have that (3.12) implies \( Y_{21} = A_{12}^{-1} \) only if \( A_{14} = 0 \) which implies the invertibility of \( A_{24} \). Hence

\[
A = \begin{bmatrix}
0 & A_{12} & 0 & 0 \\
0 & 0 & 0 & A_{24} \\
0 & 0 & 0 & 0 \\
\end{bmatrix} : \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3 \\
\mathcal{H}_4 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{N}(A^*) \\
\end{bmatrix}.
\]

It is easy to get \( \mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus [\mathcal{R}(B) \cap \mathcal{N}(A)] \).

(iii)\(\Rightarrow\)(i) Since \( \mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus [\mathcal{R}(B) \cap \mathcal{N}(A)] \) is equivalent to \( A_{14} = 0 \), we obtain from (3.12) that \( Y_{21} = A_{12}^{-1}, Y_{22} = 0, Y_{12} = 0 \). Hence, \( B^\dagger A^\dagger \in (AB)\{1, 2, 3\} \).

(i)\(\Leftrightarrow\)(ii) Using the representation of arbitrary \( X \in B\{1, 2, 3\} \) given by (3.2), we get that \(XA^\dagger \in (AB)\{1, 2, 3\} \) if and only if \( B^\dagger A^\dagger \in (AB)\{1, 2, 3\} \).

(2) If \( N(AB) = N(B) \), the proof is analogous to the case (1).

\( \Box \)

The case \( K = \{1, 2, 4\} \) is treated completely analogously, and the corresponding result follows by taking adjoints, or by reversal of products:

**Theorem 3.4.** Let \( \mathcal{H}, \mathcal{K} \) and \( \mathcal{L} \) be Hilbert spaces and let \( A \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \), \( B \in \mathcal{B}(\mathcal{L}, \mathcal{K}) \) be such that \( \mathcal{R}(A), \mathcal{R}(B), \mathcal{R}(AB) \) are closed and \( AB \neq 0 \). Then the following statements are equivalent:

(i) \( B^\dagger A^\dagger \in (AB)\{1, 2, 4\} \).

(ii) \( B^\dagger A\{1, 2, 4\} \subseteq (AB)\{1, 2, 4\} \).

(iii) \( \mathcal{R}(BB^*A^*) = \mathcal{R}(A^*) \ominus [\mathcal{R}(A^*) \cap \mathcal{N}(B^*)] \).

From the above two theorems, we get the following equivalent condition for the reverse order law for the Moore-Penrose inverse.
**Theorem 3.5.** Let $H$, $K$ and $L$ be Hilbert spaces and let $A \in \mathcal{B}(H,K)$, $B \in \mathcal{B}(L,H)$ be such that $\mathcal{R}(A), \mathcal{R}(B), \mathcal{R}(AB)$ are closed and $AB \neq 0$. Then the following statements are equivalent:

(i) $(AB)^\dagger = B^\dagger A^\dagger$.

(ii) $\mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus \perp [\mathcal{R}(B) \cap \mathcal{N}(A)]$ and $\mathcal{R}(BB^*A^*) = \mathcal{R}(A^*) \ominus \perp [\mathcal{R}(A^*) \cap \mathcal{N}(B^*)]$.

**Remark 3.4.** The conditions (ii) from Theorem 3.5 are equivalent with the conditions $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$ given in the paper by Greville [8] for matrices. Also, they are equivalent to those given in the Theorem 2.2 (c) [5] in the case of bounded operators on the Hilbert space.

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**References**


