Mixed-type reverse-order laws for \{1,3,4\}-generalized inverses over Hilbert spaces

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\textbf{Abstract}

The reverse order laws for \{1,3,4\}-generalized inverses of a product of two operators have been studied by Wang et al. [J. Wang, H. Zhang, G. Ji, A generalized reverse order law for the products of two operators, Journal of Shaanxi Normal University, 38 (4) (2010), 13–17]. In this paper using a block-operator matrix technique we study mixed-type reverse-order laws for \{1,3,4\}-generalized inverses over Hilbert spaces.

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\section{Introduction}

Let \(\mathcal{H}\) and \(\mathcal{K}\) be complex Hilbert spaces. By \(\mathcal{L}(\mathcal{H}, \mathcal{K})\) we denote the set of bounded linear operators from \(\mathcal{H}\) to \(\mathcal{K}\). For a given \(T \in \mathcal{L}(\mathcal{H}, \mathcal{K})\), by \(\mathcal{N}(T)\) and \(\mathcal{R}(T)\) we denote the kernel of \(T\) and the range of \(T\), respectively.

Recall that \(A \in \mathcal{L}(\mathcal{H}, \mathcal{K})\) has a Moore–Penrose inverse if there exists an operator \(G \in \mathcal{L}(\mathcal{K}, \mathcal{H})\) such that

\[
\begin{align*}
(1) & \quad AGA = A \\
(2) & \quad GAG = G \\
(3) & \quad (AG)^\dagger = AG \\
(4) & \quad (GA)^\dagger = GA
\end{align*}
\]

Equations (1.1) are called the \(\{1,3,4\}\)-reverse order laws for \(A\). If one of the above equations holds for an \(A \in \mathcal{L}(\mathcal{H}, \mathcal{K})\), we call \(A\) a \(\{1,3,4\}\)-inverse of \(A\).

In [17–21], Greville [8] proved that \((AB)^\dagger = B^\dagger A^\dagger\) if and only if \(\mathcal{R}(A^\dagger AB) \subseteq \mathcal{R}(B)\) and \(\mathcal{R}(BB^\dagger A^\dagger) \subseteq \mathcal{R}(A^\dagger)\), for matrices \(A\) and \(B\). This result was extended to linear bounded operators on Hilbert spaces in [10]. Later, the reverse order law for the Moore–Penrose inverse was considered in rings with involution. Werner [18] presented conditions for the inclusion \(B^\dagger A^\dagger \subseteq (AB)^\dagger\) to hold. Wei [19,20], Wei and Guo [21] studied reverse-order laws for \{1\}-inverses, \{1,3\}-inverses and \{1,4\}-inverses of matrix products. In [4], the reverse order laws for \{1,3\}, \{1,4\}, \{1,2,3\}, \{1,2,4\}-inverses were considered while in [5] the reverse order law for \{1,3,4\}-inverse in \(C^\ast\)-algebras was investigated. For other interesting results on this subject see [1,2,6,9,11–16,22–26]. In this paper, using a block-operator matrix technique, we obtain necessary and sufficient conditions for the mixed-type reverse-order laws for \{1,3,4\}-generalized inverses over Hilbert spaces.

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2. Preliminaries

Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ have closed range. It is well-known that the operator $A$ has the following decomposition (see, e.g., [7] or [3]).

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix},$$

(2.1)

where $A_1$ is an invertible. In that case, $A^1$ can be represented by

$$A^1 = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^1) \\ \mathcal{N}(A) \end{bmatrix}.$$  

(2.2)

The following lemma will be useful:

**Lemma 2.1.** Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. If $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ has closed range, then

(i) $A(1, 3) = \{A^1 + (I - A^1)X : X \in \mathcal{L}(\mathcal{K}, \mathcal{H})\}$,

(ii) $A(1, 4) = \{A^1 + Y(I - AA^1) : Y \in \mathcal{L}(\mathcal{K}, \mathcal{H})\}$.

If $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ has closed range, then we can explicitly describe the sets $A(1, 3), A(1, 4), A(1, 2, 3), A(1, 2, 4)$ and $A(1, 3, 4)$ using the representation of $A$ given by (2.1).

**Lemma 2.2.** Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ be given by (2.1). If $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ has closed range, then

$$A(1, 3) = \begin{bmatrix} A_1^{-1} & 0 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^1) \\ \mathcal{N}(A) \end{bmatrix},$$

$$A(1, 4) = \begin{bmatrix} A_1^{-1} & X_2 \\ 0 & X_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^1) \\ \mathcal{N}(A) \end{bmatrix},$$

$$A(1, 2, 3) = \begin{bmatrix} A_1^{-1} & 0 \\ X_3 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^1) \\ \mathcal{N}(A) \end{bmatrix},$$

$$A(1, 2, 4) = \begin{bmatrix} A_1^{-1} & X_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^1) \\ \mathcal{N}(A) \end{bmatrix},$$

and

$$A(1, 3, 4) = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & X_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^1) \\ \mathcal{N}(A) \end{bmatrix},$$

where $X_2, X_3$, and $X_4$ are arbitrary bounded linear operators defined on appropriate subspaces.

From the following lemma, which was proved in [17] we get a useful representation of two operators $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and $B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$.

**Lemma 2.3** [17]. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K}), B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B), \mathcal{R}(AB)$ are closed. Let

$$\mathcal{H}_1 = \mathcal{R}(B) \cap \mathcal{N}(A),$$

$$\mathcal{H}_2 = \mathcal{R}(B) \oplus \mathcal{H}_1,$$

$$\mathcal{H}_3 = \mathcal{N}(B^*) \cap \mathcal{N}(A),$$

$$\mathcal{H}_4 = \mathcal{N}(B^*) \oplus \mathcal{H}_3.$$  

Then $A$ and $B$ have the following operator matrix forms

$$A = \begin{bmatrix} 0 & A_{12} & 0 & A_{14} \\ 0 & 0 & A_{24} \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \end{bmatrix}.$$  

(2.3)
and

\[
B = \begin{bmatrix}
B_{11} & B_{12} & 0 \\
0 & B_{22} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} : \begin{bmatrix}
\mathcal{J}_1 \\
\mathcal{J}_2 \\
\mathcal{N}(B)
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix},
\]

(2.4)

where \(A_{12}, B_{11}, B_{22}\) are invertible and \(A_{24}\) is a surjection.

**Lemma 2.4.** Let \(\mathcal{H}\) and \(\mathcal{K}\) be Hilbert spaces and let \(A \in \mathcal{L}(\mathcal{H}, \mathcal{K}), B \in \mathcal{L}(\mathcal{K}, \mathcal{H})\) be such that \(\mathcal{R}(A), \mathcal{R}(B)\) and \(\mathcal{R}(AB)\) are closed. If \(A\) and \(B\) are given by (2.3) and (2.4) respectively, then the set \(A\{1, 3, 4\}\) consists of all operator matrices \(X\) that can be represented by

\[
X = \begin{bmatrix}
0 & 0 & X_{13} \\
X_{21} & X_{22} & X_{23} \\
0 & 0 & X_{33} \\
X_{41} & X_{42} & X_{43}
\end{bmatrix} : \begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{N}(A')
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix},
\]

(2.5)

where

\[
\begin{align*}
A_{12}X_{21} + A_{14}X_{41} &= 1, \\
A_{24}X_{41} &= 0, A_{24}X_{43} = 0, \\
A_{12}X_{23} + A_{14}X_{43} &= 0, \\
A_{12}X_{22} + A_{14}X_{42} &= 0, \\
A_{24}X_{42} &= I, \\
(X_{21}A_{14} + X_{22}A_{24})' &= X_{41}A_{12}
\end{align*}
\]

and

\[
X_{23}A_{12}, X_{41}A_{14} + X_{42}A_{24}
\]

are selfadjoint.

Also, the set \(B\{1, 3, 4\}\) consists of all such \(Y\) that can be represented by

\[
Y = \begin{bmatrix}
B_{11}^{-1} & -B_{11}^{-1}B_{12}B_{22}^{-1} & 0 & 0 \\
0 & B_{22}^{-1} & 0 & 0 \\
0 & 0 & F_1 & F_2
\end{bmatrix} : \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{J}_1 \\
\mathcal{J}_2 \\
\mathcal{N}(B)
\end{bmatrix},
\]

(2.8)

where \(F_1, F_2\) are arbitrary operators defined on appropriate subspaces.

**Proof.** The part concerning \(B\) follows trivially from **Lemma 2.2** and **Lemma 2.3**. Take an arbitrary \(X \in A\{1, 3, 4\}\) and represent it by

\[
X = \begin{bmatrix}
X_{11} & X_{12} & X_{13} \\
X_{21} & X_{22} & X_{23} \\
X_{31} & X_{32} & X_{33} \\
X_{41} & X_{42} & X_{43}
\end{bmatrix} : \begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{N}(A')
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix},
\]

(2.9)

From \(XA = (XA)'\), the invertibility of \(A_{12}\) and the surjectivity of \(A_{24}\), we get that \(X_{11} = 0, X_{31} = 0, X_{12} = 0\) and \(X_{32} = 0\). Also, we get that \((X_{21}A_{14} + X_{22}A_{24})' = X_{41}A_{12}\) and that \(X_{21}A_{12}, X_{41}A_{14} + X_{42}A_{24}\) are selfadjoint.

By \(AXA = A\), we get \(A_{12}X_{21} + A_{14}X_{41} = 1, A_{24}X_{41} = 0\) and \(A_{24}X_{42} = I\). Now using \((AX)' = AX\) we obtain \(A_{12}X_{22} + A_{14}X_{42} = 0, A_{12}X_{23} + A_{14}X_{43} = 0\) and \(A_{24}X_{43} = 0\).

Remark that under the assumptions of the previous lemma, we have that \(Z \in \{AB\}\{1, 3, 4\}\) if and only if there exist \(M_1 \in \mathcal{L}(\mathcal{K}_2, \mathcal{J}_1), M_2 \in \mathcal{L}(\mathcal{N}(A'), \mathcal{J}_1), M_3 \in \mathcal{L}(\mathcal{K}_2, \mathcal{N}(B))\) and \(M_4 \in \mathcal{L}(\mathcal{N}(A'), \mathcal{N}(B))\) such that

\[
Z = \begin{bmatrix}
0 & M_1 & M_2 \\
B_{22}^{-1}A_{12} & 0 & 0 \\
0 & M_3 & M_4
\end{bmatrix} : \begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{N}(A')
\end{bmatrix} \rightarrow \begin{bmatrix}
\mathcal{J}_1 \\
\mathcal{J}_2 \\
\mathcal{N}(B)
\end{bmatrix},
\]

(2.10)

**Lemma 2.5.** Let \(\mathcal{H}\) and \(\mathcal{K}\) be Hilbert spaces and let \(A \in \mathcal{L}(\mathcal{H}, \mathcal{K}), B \in \mathcal{L}(\mathcal{K}, \mathcal{H})\) be such that \(\mathcal{R}(A), \mathcal{R}(B)\) and \(\mathcal{R}(AB)\) are closed. If \(B\) is given by (2.4) then \(B_{12} = 0\) if and only if \(B'(\mathcal{R}(B) \cap \mathcal{N}(A)) \subseteq B'(\mathcal{R}(B) \cap \mathcal{N}(A))\).
Suppose that operators $B_1$ and $B_2$ belong to $\mathcal{H}_1$. By Lemma 2.4, Theorem 3.1.

\begin{align*}
\mathcal{J}_1 & = \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{N}(B) \end{pmatrix}, \\
\mathcal{J}_2 & = \begin{pmatrix} \mathcal{J}_1 \\ \mathcal{N}(B) \end{pmatrix}.
\end{align*}

Now, we have

$$B_{12} = 0 \iff B_{12}^* = 0 \iff B' \mathcal{H}_1 \subseteq \mathcal{J}_1.$$

By definition of $\mathcal{H}_1$ and $\mathcal{J}_1$, we have $B_{12} = 0$ if and only if $B' (\mathcal{R}(B) \cap \mathcal{N}(A)) \subseteq B' (\mathcal{R}(B) \cap \mathcal{N}(A))$. $\square$

3. Mixed-type reverse-order law of $\{1, 3, 4\}$–inverses

In this section we will give necessary and sufficient conditions for the reverse-order laws

$$B(1, 3, 4) \left( ABB^{(1,3,4)} \right)(1, 3, 4) \subseteq (AB)(1, 3, 4), \text{ for any } B^{(1,3,4)} \in B(1, 3, 4)$$

and

$$B(1, 3, 4)A(1, 3, 4) = (AB)(1, 3, 4).$$

**Theorem 3.1.** em Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ be such that $\mathcal{R}(A)$, $\mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed. The following statements are equivalent:

1. $B(1, 3, 4) \left( ABB^{(1,3,4)} \right)(1, 3, 4) \subseteq (AB)(1, 3, 4)$, for any $B^{(1,3,4)} \in B(1, 3, 4)$,
2. $B' (\mathcal{R}(B) \cap \mathcal{N}(A)) \subseteq B' (\mathcal{R}(B) \cap \mathcal{N}(A))$.

**Proof.** Suppose that operators $A$ and $B$ are given by (2.3) and (2.4), respectively. We assume that $\mathcal{H}_i, \mathcal{K}_j, \mathcal{J}_j \neq 0$ for $i = 1, 4$, $j = 1, 2$. In other cases, the proof is similar.

(i) $\Rightarrow$ (ii): Let $Y \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ be defined by

$$Y = \begin{bmatrix}
B_{11} & -B_{12}^*B_{22}^* & 0 & 0 \\
0 & B_{22}^* & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
: \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\mathcal{J}_1 \\
\mathcal{J}_2 \\
\mathcal{N}(B)
\end{bmatrix}.$$

By Lemma 2.4, $Y \in B(1, 3, 4)$. By simple computation we get that

$$ABY = \begin{bmatrix}
0 & A_{12} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
: \begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{N}(A')
\end{bmatrix}.$$

Since the operator $Z$ defined by

$$Z = \begin{bmatrix}
A_{12} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
: \begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{N}(A')
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\mathcal{H}_1 \\
\mathcal{H}_2 \\
\mathcal{H}_3 \\
\mathcal{H}_4
\end{bmatrix}.$$

belongs to $(AB)(1, 3, 4)$, by (i) we have that $YZ \in (AB)(1, 3, 4)$. From

$$YZ = \begin{bmatrix}
-B_{11}^*B_{12}B_{22}A_{12}^* & 0 & 0 \\
B_{22}^*A_{12}^* & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
: \begin{bmatrix}
\mathcal{K}_1 \\
\mathcal{K}_2 \\
\mathcal{N}(A')
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\mathcal{J}_1 \\
\mathcal{J}_2 \\
\mathcal{N}(B)
\end{bmatrix}.$$

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using (2.10) we get $B_{12} = 0$. Now (ii) follows by Lemma 2.5.

(ii) $\Rightarrow$ (i) Suppose that (ii) holds. Then

$$\begin{align*}
B &= \begin{bmatrix} B_{11} & 0 & 0 \\
0 & B_{22} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} : \begin{bmatrix} J_1 \\
J_2 \\
N(B) \\
\end{bmatrix} \rightarrow \begin{bmatrix} H_1 \\
H_2 \\
H_3 \\
H_4 \\
\end{bmatrix}.
\end{align*}$$

(3.5)

Let $Y_i \in B\{1, 3, 4\}$ be arbitrary and take any $S \in B\{1, 3, 4\}(ABY_1)\{1, 3, 4\}$. There exists $Y_2 \in B\{1, 3, 4\}$ such that $S \in Y_2(ABY_1)\{1, 3, 4\}$. By Lemma 2.4, we have that $Y_1, Y_2$ have the form

$$Y_i = \begin{bmatrix} B_{11}^{-1} & 0 & 0 & 0 \\
0 & B_{22}^{-1} & 0 & 0 \\
0 & 0 & F_{1i} & F_{2i} \\
\end{bmatrix} : \begin{bmatrix} H_1 \\
H_2 \\
H_3 \\
H_4 \\
\end{bmatrix} \rightarrow \begin{bmatrix} J_1 \\
J_2 \\
N(B) \\
\end{bmatrix}, \quad i = 1, 2,$n

(3.6)

for some operators $F_{1i}, F_{2i}, i = 1, 2$. Now

$$ABY_1 = \begin{bmatrix} 0 & A_{12} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} : \begin{bmatrix} H_1 \\
H_2 \\
H_3 \\
H_4 \\
\end{bmatrix} \rightarrow \begin{bmatrix} K_1 \\
K_2 \\
N(A') \\
\end{bmatrix}$$

and $Z \in (ABY_1)\{1, 3, 4\}$ if and only if there exist operators $K_i, i = 1, 2, 3, 4$ such that

$$Z = \begin{bmatrix} 0 & K_1 & K_2 \\
A_{12}^{-1} & 0 & 0 \\
0 & K_3 & K_4 \\
0 & K_5 & K_6 \\
\end{bmatrix} : \begin{bmatrix} K_1 \\
K_2 \\
N(A') \\
\end{bmatrix} \rightarrow \begin{bmatrix} H_1 \\
H_2 \\
H_3 \\
H_4 \\
\end{bmatrix}.$$n

(3.7)

Hence, arbitrary $S \in Y_2(ABY_1)\{1, 3, 4\}$ has the form

$$S = \begin{bmatrix} 0 & B_{11}^{-1}K_1 & B_{11}^{-1}K_2 \\
B_{22}^{-1}A_{12}^{-1} & 0 & 0 \\
0 & F_{1i}K_3 + F_{2i}K_5 & F_{1i}K_4 + F_{2i}K_6 \\
\end{bmatrix} : \begin{bmatrix} K_1 \\
K_2 \\
N(A') \\
\end{bmatrix} \rightarrow \begin{bmatrix} J_1 \\
J_2 \\
N(B) \\
\end{bmatrix},$$

for some operators $K_i, i = 1, 2, 3, 4.$

Now, by (2.10) we get $S \in (AB)\{1, 3, 4\}$. □

**Theorem 3.2.** Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed. The following statements are equivalent:

(i) $ABB\{1, 3, 4\}A\{1, 3, 4\}AB = AB$,

(ii) $B'A\{1, 3, 4\}AB = B'(ABB^{(1,3,4)})AB$, for any $B^{(1,3,4)} \in B\{1, 3, 4\}$.

**Proof.** Suppose that operators $A$ and $B$ are given by (2.3) and (2.4), respectively. We assume that $\mathcal{H}_i, \mathcal{K}_i, J_i \neq 0$ for $i = 1, 2$. In other cases, the proof is similar.

Pick arbitrary $X_1, X_2, Y \in B\{1, 3, 4\}$. By Lemma 2.4 we have that $X, Y$ are represented by (2.5) and (2.8) respectively. From (i) we have that $ABXYAB = AB$. Remark that

$$ABXYAB = \begin{bmatrix} 0 & A_{12}X_{21}A_{12}B_{22} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} : \begin{bmatrix} J_1 \\
J_2 \\
N(B) \\
\end{bmatrix} \rightarrow \begin{bmatrix} K_1 \\
K_2 \\
N(A') \\
\end{bmatrix}.$$n

(3.8)

and

$$AB = \begin{bmatrix} 0 & A_{12}B_{22} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} : \begin{bmatrix} J_1 \\
J_2 \\
N(B) \\
\end{bmatrix} \rightarrow \begin{bmatrix} K_1 \\
K_2 \\
N(A') \\
\end{bmatrix}.$$n

(3.9)

From (3.8) and (3.9) and the invertibility of $A_{12}$ and $B_{22}$, we get $ABB\{1, 3, 4\}A\{1, 3, 4\}AB = AB$ if and only if $X_{21} = A_{12}^{-1}$. 


On the other hand,
\[
B' X A B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & B_{22} X A_1 & A_2 B_{22} \\ 0 & 0 & 0 \end{bmatrix}.
\] (3.10)

and for any \(B^{(1,3,4)} \in B(1,3,4)\),
\[
B'(ABB^{(1,3,4)}) A B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & B_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\] (3.11)

Now, from (3.10), (3.11) and the invertibility of \(B_{22}\), we get \(B' X A B = B'(ABB^{(1,3,4)}) A B\), for any \(B^{(1,3,4)} \in B(1,3,4)\) if and only if \(X_{21} = A_{12}^{-1}\). Hence, (i) \(\iff\) (ii).

**Theorem 3.3.** Let \(\mathcal{H}\) and \(\mathcal{K}\) be Hilbert spaces and let \(A \in \mathcal{L}(\mathcal{H}, \mathcal{K}), B \in \mathcal{L}(\mathcal{K}, \mathcal{H})\) be such that \(\mathcal{R}(A), \mathcal{R}(B)\) and \(\mathcal{R}(AB)\) are closed. The following statements are equivalent:

1. \((AB)(1,3,4)A = B(1,3,4)A(1,3,4)AB,\)
2. \(ABB'B(1,3,4) = ABB'A(1,3,4)ABB(1,3,4)\) and \(\mathcal{R}(ZAB) = \mathcal{R}(YXAB)\), for any \(Z \in (AB)(1,3,4), X \in A(1,3,4), Y \in B(1,3,4).\)

**Proof.** Suppose that operators \(A\) and \(B\) are given by (2.3) and (2.4), respectively. We assume that \(\mathcal{H}_i, \mathcal{K}_j, \mathcal{J}_j \neq 0\) for \(i = 1, 2, 3, 4\), \(j = 1, 2\). In other cases, the proof is similar. Pick arbitrary \(X \in A(1,3,4), Y \in B(1,3,4)\) and \(Z \in (AB)(1,3,4)\). By computation, we get
\[
Z A B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]
and
\[
Y X A B = \begin{bmatrix} 0 & -B_{11}^{-1} B_{12} X_{21} A_1 & A_2 B_{22} \\ 0 & B_{21}^{-1} X_{21} A_2 B_{12} & 0 \\ 0 & F_2 X_{41} A_1 A_2 B_{22} & 0 \end{bmatrix}.
\] (3.12)

Also,
\[
A B' B Y = \begin{bmatrix} A_1 B_{22} B_1' B_2 & A_1 B_{22} B_{22}' \end{bmatrix}
\]
and
\[
A B' X A B Y = \begin{bmatrix} 0 & A_1 B_{22} B_{22}' X_{21} A_1 \end{bmatrix}
\]

(i) \(\Rightarrow\) (ii): If (i) holds, then by (3.12) we have \(B_{12} = 0, X_{21} = A_{12}^{-1}\) and \(X_{41} = 0\). So (ii) holds.

(ii) \(\Rightarrow\) (i): From \(ABB'B(1,3,4) = ABB'A(1,3,4)ABB(1,3,4)\) we obtain \(B_{12} = 0\) and \(X_{21} = A_{12}^{-1}\). By \(\mathcal{R}(ZAB) = \mathcal{R}(YXAB)\) we get \(F_2 X_{41} = 0\). Since \(F_2\) is arbitrary, it follows \(X_{41} = 0\). Hence, (i) holds.

**Theorem 3.4.** Let \(\mathcal{H}\) and \(\mathcal{K}\) be Hilbert spaces and let \(A \in \mathcal{L}(\mathcal{H}, \mathcal{K}), B \in \mathcal{L}(\mathcal{K}, \mathcal{H})\) be such that \(\mathcal{R}(A), \mathcal{R}(B)\) and \(\mathcal{R}(AB)\) are closed. The following statements are equivalent:

1. \(B' A(1,3,4) = (AB)\)
2. \(B' X \in B'(AB)(1,3,4) = (AB)(1,3,4)\) and \(\mathcal{R}(BB' X) = \mathcal{R}((AB)^*)\), for any \(X \in A(1,3,4), Y \in B(1,3,4)\).

**Proof.** Pick arbitrary \(X \in A(1,3,4), Y \in B(1,3,4)\). By Lemma 2.4 we have that \(X, Y\) are represented by (2.5) and (2.8) respectively. Since
\[
A B Y = \begin{bmatrix} 0 & A_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]
we have that any $Z \in (ABY)\{1, 3, 4\}$ has the matrix form

$$Z = \begin{bmatrix} 0 & K_1 & K_2 \\ A_{12}^{-1} & 0 & 0 \\ 0 & K_3 & K_4 \\ 0 & K_5 & K_6 \end{bmatrix}$$

for some operators $K_i$, $i = 1, 6$.

From Lemma 2.4 we get

$$B^i Z = \begin{bmatrix} -B_{11}^{-1}B_{12}B_{22}^{-1}A_{12}^{-1} & B_{11}^{-1}K_1 & B_{11}^{-1}K_2 \\ B_{22}^{-1}A_{12}^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and for arbitrary $X \in A\{1, 3, 4\}$,

$$B^i X = \begin{bmatrix} -B_{11}^{-1}B_{12}B_{22}^{-1}X_{21} & -B_{11}^{-1}B_{12}B_{22}^{-1}X_{22} & -B_{11}^{-1}B_{12}B_{22}^{-1}X_{23} \\ B_{22}^{-1}X_{21} & B_{22}^{-1}X_{22} & B_{22}^{-1}X_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

(i) $\Rightarrow$ (ii): From $B^i A\{1, 3, 4\} = (AB)^i$ it follows that $X_{21} = A_{12}^{-1}$, $X_{13} = 0$, $X_{22} = 0$, $X_{23} = 0$ and $B_{12} = 0$. Hence, from (3.13) and (3.14), we get that (ii) holds.

(ii) $\Rightarrow$ (i): If (ii) holds, we get $X_{21} = A_{12}^{-1}$, $X_{22} = 0$, $X_{23} = 0$ and $B_{12} = 0$. Hence, we have that

$$B^i X = \begin{bmatrix} 0 & 0 & B_{11}^{-1}X_{13} \\ B_{22}^{-1}A_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  

(3.15)

Since

$$BB^i X = \begin{bmatrix} 0 & 0 & X_{13} \\ A_{12}^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $(ABY)^i = \begin{bmatrix} 0 & 0 & 0 \\ A_{12}^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

from $\mathcal{R}(BB^i X) = \mathcal{R}((ABY)^i)$, we conclude $X_{13} = 0$. Hence, (i) holds.

**Theorem 3.5.** Let $H$ and $K$ be Hilbert spaces and let $A \in \mathcal{L}(H,K)$, $B \in \mathcal{L}(K,H)$ be such that $\mathcal{R}(A)$, $\mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed. The following statements are equivalent:

(i) $(AB)^i \{1, 3, 4\} \subseteq (B\{1, 3, 4\})^i$,

(ii) $B^i (R(B) \cap N(A)) \subseteq B^i (R(B) \cap N(A))$.

**Proof.** Suppose that operators $A$ and $B$ are given by (2.3) and (2.4), respectively. We assume that $\mathcal{H}_i, \mathcal{K}_j, \mathcal{J}_j \neq 0$ for $i = 1, 4$, $j = 1, 2$. In other cases, the proof is similar.

By (2.8) and (2.10), for arbitrary $B^{i,1,3,4} \in B\{1, 3, 4\}$, we get

$$(AB)^i \{1, 3, 4\} = \begin{bmatrix} 0 & 0 & 0 \\ B_{11}^{-1}B_{12} & 0 & 0 \\ 0 & F_1 & F_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ B_{22}^{-1}A_{12}^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(3.17)

By (2.3) and (2.4), we have

$$BAB = \begin{bmatrix} 0 & B_{11}A_{12}B_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$  

(3.18)
Therefore, arbitrary \((BAB)^{(1,3,4)} \in (BAB)\{1,3,4\}\) is given by
\[
(BAB)^{(1,3,4)} = \begin{bmatrix}
0 & L_1 & L_2 \\
B_{12}^{-1}A_{12}^{-1} & 0 & 0 \\
0 & L_4 & L_5 \\
\end{bmatrix}
\]
for some operators \(L_i, i = 1,6\).

Hence, from (3.17), (3.19) and Lemma 2.5, we get \((AB)^\dagger B\{1,3,4\} \subseteq (BAB)\{1,3,4\}\) if and only if \(B_{12} = 0\) if and only if \(B^\dagger (\mathcal{R}(B) \cap \mathcal{N}(A)) \subseteq B^{\dagger}\).

Now, from Theorem 3.1 and Theorem 3.5, we get the following corollary:

**Corollary 3.1.** Let \(\mathcal{H}\) and \(\mathcal{K}\) be Hilbert spaces and let \(A \in \mathcal{L}(\mathcal{H}, \mathcal{K})\), \(B \in \mathcal{L}(\mathcal{K}, \mathcal{H})\) be such that \(\mathcal{R}(A), \mathcal{R}(B)\) and \(\mathcal{R}(AB)\) are closed. The following statements are equivalent:

1. \(B\{1,3,4\} (AB)^{(1,3,4)} \subseteq (BAB)\{1,3,4\}\), for any \(B^{(1,3,4)} \in B\{1,3,4\}\),
2. \((AB)^\dagger B\{1,3,4\} \subseteq (BAB)\{1,3,4\}\).

**References**

22. H.J. Werner, When is \(B^{\dagger}A\) a generalized inverse of \(AB^\dagger\)? Linear Algebra Appl. 210 (1994) 255–263.