On a problem of N. Thome and Y. Wei

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Abstract

In this note we solve a recent problem posed by N. Thome and Y. Wei (Appl. Math. Comput. 141, (2003)).

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Let $C^{m \times n}$ denote the set of complex $m \times n$ matrices. If $A \in C^{n \times n}$, the matrix $X \in C^{n \times n}$ which satisfies the following

$$A^k X A = A^k, \quad XAX = X, \quad AX =XA,$$

is called the Drazin inverse of $A$ and it is denoted by $A^d$. The smallest number $k$ such that there exist $X$ which satisfies (1) is called the index of $A$ and it is denote by $\text{ind}(A)$.

Let us recall that N. Thome and Y. Wei [1] have proved the following theorem:

**Theorem 1** Let $A \in C^{n \times n}$ with $\text{ind}(A) = 1$, $\text{rank}(A) = r$, $B, C$ and $X \in C^{n \times n}$. The matrix $A$ can be written as

$$A = P^{-1} \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} P,$$

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where $P \in C^{n \times n}, M \in C^{r \times r}$ are nonsingular matrices and $M$ is upper bidiagonal. Then $X = A^d$ is the solution of the equation

$$\operatorname{rank} \begin{bmatrix} A & B \\ C & X \end{bmatrix} = \operatorname{rank}(A),$$

if and only if

$$B = P^{-1} \begin{bmatrix} MG_1 & 0 \\ 0 & 0 \end{bmatrix} P \text{ and } C = P^{-1} \begin{bmatrix} M^{-1}G_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} P$$

for some nonsingular matrix $G_1 \in C^{r \times r}$.

They set the following question: Is it possible to extend Theorem 1 in the case when $\text{ind}(A) = k > 1$? In this note we solved that question.

Let us recall that the Moore-Penrose inverse of $A \in C^{n \times m}$ is the unique matrix $A^\dagger \in C^{m \times n}$ which satisfies

$$AA^\dagger A = A, \ A^\dagger AA^\dagger = A^\dagger, \ (AA^\dagger)^* = AA^\dagger, \ (A^\dagger A)^* = A^\dagger A$$

and the inner inverse of $A$ is the matrix $A^-$ which satisfies $AA^-A = A$. By $A^*$, $R(A)$ and $\operatorname{rank}(A)$ we denote the conjugate transpose, the range and the rank of $A \in C^{n \times m}$.

Notice that the conditions $R(B) \subseteq R(A)$ and $R(C^\ast) \subseteq R(A^\ast)$ are equivalent to $AA^\dagger B = B$ and $CA^\dagger A = C$. Furthermore, matrix product $CA^\dagger B$ is invariant with respect to the choice of generalized inverse $A^-$ of $A$ if and only if $R(B) \subseteq R(A)$ and $R(C^\ast) \subseteq R(A^\ast)$.

Now, we give our main result:

**Theorem 2** Let $A \in C^{n \times n}, \ ind(A) = k, \ \operatorname{rank}(A^k) = r$ and $B, C, X \in C^{n \times n}$. The matrix $A$ can be written as

$$A = P^{-1} \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} P,$$

where $P \in C^{n \times n}, M \in C^{r \times r}$ are nonsingular and $N \in C^{(n-r) \times (n-r)}$ is nilpotent. Then $X = A^d$ is a solution of the equation (3) if and only if there exist $G_1, F_1 \in C^{r \times r}, \ G_2, F_2 \in C^{r \times (n-r)}, \ G_3, F_3 \in C^{(n-r) \times r},$ and $G_4, F_4 \in C^{(n-r) \times (n-r)}$ such that

$$B = P^{-1} \begin{bmatrix} MG_1 & MG_2 \\ NG_3 & NG_4 \end{bmatrix} P \text{ and } C = P^{-1} \begin{bmatrix} F_1M & F_2N \\ F_3M & F_4N \end{bmatrix} P$$

(6)
and
\[ F_1 MG_1 + F_2 NG_3 = M^{-1}, \tag{7} \]
\[ F_1 MG_2 + F_2 NG_4 = 0, \]
\[ F_3 MG_1 + F_4 NG_3 = 0, \]
\[ F_3 MG_2 + F_4 NG_4 = 0. \]

**Proof.** Suppose that \( X = A^d \) is a solution of the equation (3). By \([2, \text{Theorem 1}]\) we have that \( R(B) \subseteq R(A) \) and \( R(C^*) \subseteq R(A^*) \), so there exist matrices \( G \) and \( F \) such that \( B = AG \), \( C = FA \) and \( A^d = CA^*B \). Let
\[ PGP^{-1} = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \quad \text{and} \quad PP^{-1} = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix}. \]

It follows that
\[ B = P^{-1} \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} P = P^{-1} \begin{bmatrix} MG_1 & MG_2 \\ NG_3 & NG_4 \end{bmatrix} P \]
and
\[ C = P^{-1} \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} P = P^{-1} \begin{bmatrix} F_1 M & F_2 N \\ F_3 M & F_4 N \end{bmatrix} P. \]

Also, we know that for matrix \( A \) which has the form (5),
\[ A^d = P^{-1} \begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} P \quad \text{and} \quad A^* = P^{-1} \begin{bmatrix} M^{-1} & 0 \\ 0 & N^{-1} \end{bmatrix} P. \]

Hence,
\[ A^d = P^{-1} \begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} P = \]
\[ P^{-1} \begin{bmatrix} F_1 M & F_2 N \\ F_3 M & F_4 N \end{bmatrix} \begin{bmatrix} M^{-1} & 0 \\ 0 & N^{-1} \end{bmatrix} \begin{bmatrix} MG_1 & MG_2 \\ NG_3 & NG_4 \end{bmatrix} P = \]
\[ P^{-1} \begin{bmatrix} F_1 MG_1 + F_2 NG_3 & F_1 MG_2 + F_2 NG_4 \\ F_3 MG_1 + F_4 NG_3 & F_3 MG_2 + F_4 NG_4 \end{bmatrix} P. \]

So, we obtain the following system
\[ F_1 MG_1 + F_2 NG_3 = M^{-1}, \]
\[ F_1 MG_2 + F_2 NG_4 = 0, \]
\[ F_3 MG_1 + F_4 NG_3 = 0, \]
\[ F_3 MG_2 + F_4 NG_4 = 0. \]
Conversly, suppose that $B$ and $C$ satisfy (6). Then $R(B) \subseteq R(A)$ and $R(C^*) \subseteq R(A^*)$ and by [2], Theorem 1 we have that there exist a solution $X = CA^{-B}$ of the equation (3). Thus (7) implies that $CA^{-B} = A^d$, that is $X = A^d$ is a solution of the equation (3).

Now, we obtain Theorem 1 as a corollary of Theorem 2.

**Proof of Theorem 1.** If we assume that $X = A^d$ is the solution of (3), then by Theorem 2 it follows that

$$B = P^{-1}\begin{bmatrix} MG_1 & MG_2 \\ 0 & 0 \end{bmatrix}P, \quad C = P^{-1}\begin{bmatrix} F_1M & 0 \\ F_3M & 0 \end{bmatrix}P,$$

and

$$F_1MG_1 = M^{-1},$$

$$F_1MG_2 = 0,$$

$$F_3MG_1 = 0,$$

$$F_3MG_2 = 0.$$  

From $F_1MG_1 = M^{-1}$ we obtain that $F_1$ and $G_1$ are nonsingular matrices and $F_1M = M^{-1}G_1^{-1}$. Now, from $F_1MG_2 = 0$ and $F_3MG_1 = 0$, we have that $G_2 = F_3 = 0$. Hence, we obtain (4). Conversely, if we assume that $B$ and $C$ have the form (4), then for $G_2 = F_3 = 0$ and $F_1 = M^{-1}G_1^{-1}M^{-1}$ we verify that the conditions (6) and (7) hold. Hence $X = A^d$ is the solution of the equation (3).

**References**
