Comments on some recent results concerning \{2, 3\} and \{2, 4\}-generalized inverses

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Abstract

We give a comment on some recent results concerning the representations of generalized \{2, 3\} and \{2, 4\}-inverses. Shorter proofs of some previous results are presented.

AMS Subj. Class.: 15A09.

Key words: generalized inverse, \{2, 3\}-generalized inverses, \{2, 4\}-generalized inverses

1 Comments

Let \( \mathbb{C}^{m \times n} \) and \( \mathbb{C}^{m \times n}_{r} \) denote the set of all complex \( m \times n \) matrices and all complex \( m \times n \) matrices of rank \( r \), respectively. \( I_{n} \) denotes the unit matrix of order \( n \). By \( A^* \), \( \text{rank}(A) \), \( \mathcal{R}(A) \) and \( \mathcal{N}(A) \) we denote the conjugate transpose, the rank, the range and the null space of \( A \in \mathbb{C}^{m \times n} \). By \( P_{M,N} \) we denote an idempotent matrix with the range \( M \) and the null-space \( N \).

Let us recall that the Moore-Penrose inverse of \( A \in \mathbb{C}^{m \times n} \) is the unique matrix \( A^\dagger \in \mathbb{C}^{n \times m} \) which satisfies

\[
(1) \, AA^\dagger A = A, \quad (2) \, A^\dagger AA^\dagger = A^\dagger, \quad (3) \, (AA^\dagger)^* = AA^\dagger, \quad (4) \, (A^\dagger A)^* = A^\dagger A.
\]

For any \( A \in \mathbb{C}^{m \times n} \), let \( A\{i, j, \ldots, l\} \) denote the set of matrices \( X \in \mathbb{C}^{n \times m} \) which satisfy equations \((i), (j), \ldots, (l)\) from among the equations \((1), (2), (3), (4)\). A matrix \( X \in A\{i, j, \ldots, l\} \) is called an \{i, j, \ldots, l\}-inverse of \( A \), and also it is denoted by \( A^{(i,j,\ldots,l)} \).

The \{2\}-inverses has many applications, for example, the application in the iterative methods for solving the nonlinear equations [1, 5] and the applications to statistics [2, 3]. In particular, \{2\}-inverse play an important role in stable approximations of ill-posed problems and in linear and nonlinear problems involving rank-deficient generalized inverse [4, 8].

The generalized inverse of the special interest is \{2\}-generalized inverse with prescribed range \( T \) and null-space \( S \), which is denoted by \( A^{(2)}_{T,S} \). The reason of the importance of this inverse is in the fact

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\(^1\)Supported by Grant No. 174007 and 144011 of the Ministry of Science, Technology and Development, Republic of Serbia.
that important six kinds of generalized inverses: the Moore-Penrose inverse $A^\dagger$, the weighted Moore-Penrose inverse $A^\dagger_{M,N}$, the Drazin inverse $A^D$, the group inverse $A^g$, the Bott-Duffin inverse $A^{(+)\frac{1}{L}}$ and the generalized Bott-Duffin inverse $A^{(+)\frac{1}{L}}$ are all $\{2\}$-generalized inverses of $A$ with prescribed range and null space.

In this paper, we will consider a representation of $A^{(2,3)}_{T,S}$ and $A^{(2,4)}_{T,S}$ inverses of a given matrix $A$. This paper has been motivated by the one of Liu and Yang [7] in which the authors consider the representation of $A^{(2,3)}_{T,S}$. Here, we present much simpler proofs than those given in [7]. Also, we obtain an analogous results for the $A^{(2,4)}_{T,S}$ generalized inverse of $A$.

In the following result we give a much simpler representation of $A^{(2,3)}_{T,S}$ using the Moore-Penrose inverse than those given in Theorem 2.3[7].

**Theorem 1.1** Let $A \in \mathbb{C}^{m \times n}$. For any matrix $U \in \mathbb{C}^{n \times m}$ which satisfies that $\text{rank}(AU) = \text{rank}(U)$, we have

$$U(AU)^\dagger = A^{(2,3)}_{R(U),R(AU)^\perp}.$$

**Proof.** Denote by $X = U(AU)^\dagger$. We can check that $XAX = X$ and $(AX)^* = (AX)$. Now, we will prove that $R(X) = R(U)$ and $N(X) = R(AU)^\perp$. Since $\text{rank}(AU) = \text{rank}(U)$, the following holds

$$\text{rank}(U) \geq \text{rank}(X) \geq \text{rank}(AX) = \text{rank}(AU) = \text{rank}(U),$$

so $R(X) = R(U)$. Similarly,

$$N(X) \subseteq N(AX) = R(AU)^\perp,$$

and

$$\dim N(AX) = m - \text{rank}(AX) = m - \text{rank}(X) = \dim N(X),$$

so $N(X) = N(AX) = R(AU)^\perp$. □

**Corollary 1.1** Let $A \in \mathbb{C}^{m \times n}$ and $U \in \mathbb{C}^{n \times m}$ be such that $\text{rank}(AU) = \text{rank}(U)$ and let $P_{R(AU)}$ be orthogonal projection on $R(AU)$. Then

$$U(AU)^\dagger = U(AU)^- P_{R(AU)} = A^{(2,3)}_{R(U),R(AU)^\perp},$$

for arbitrary $(AU)^- \in (AU)\{1\}$.

**Proof.** For each $(AU)^- \in (AU)\{1\}$, we have that

$$U(AU)^- P_{R(AU)} = U(AU)^-(AU)(AU)^\dagger = UP_{R((AU)^-(AU)),N(AU)}(AU)^\dagger = UP_{R((AU)^-(AU)),N(U)(AU)}(AU)^\dagger = U(AU)^\dagger.$$

Now, the proof can be completed using the representation of $\{2,3\}$-inverses from Theorem 1.1. □

As a corollary we get the main result from [7]:
Let $A \in \mathbb{C}^{m \times n}$, $T$ and $S$ be subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, and $\dim(AT) = \dim(T)$. For any matrix $U \in \mathbb{C}^{n \times m}$ which satisfies $\mathcal{R}(U) = T$, we have

$$A_{T,S}^{(2,3)} = U(AU)^\dagger = U(AU)^\dagger P_{\mathcal{R}(AU)}$$

if and only if $\mathcal{R}(AT) = S^\perp$, where $P_{\mathcal{R}(AU)}$ is the orthogonal projection on $\mathcal{R}(AU)$. Moreover, $U(AU)^\dagger P_{\mathcal{R}(AU)}$ is independent of the choice of $(AU)^\dagger \in (AU)\{1\}$.

**Proof.** By Corollary 1.1, we have that

$$U(AU)^\dagger = U(AU)^\dagger P_{\mathcal{R}(AU)} = A_{\mathcal{R}(U),\mathcal{R}(AU)}^{(2,3)} = A_{T,\mathcal{R}(AU)}^{(2,3)}.$$ 

So, $U(AU)^\dagger = U(AU)^\dagger P_{\mathcal{R}(AU)} = A_{T,S}^{(2,3)}$ if and only if $\mathcal{R}(AU)^\dagger = S$, which is equivalent with $\mathcal{R}(AT) = S^\perp$. The fact that $U(AU)^\dagger P_{\mathcal{R}(AU)}$ is independent of the choice of $(AU)^\dagger \in (AU)\{1\}$ follows by Corollary 1.1. □

Remark that in Theorem 2.2[7] authors consider arbitrary hermitian idempotent matrix satisfying $\mathcal{R}(M) = AT$ which is incorrect because matrix $M$ with these properties is a unique. So, Theorem 2.2 and Theorem 2.3 from [7] are exactly the same.

Analogously, we get similar results for $\{2,4\}$-generalized inverses:

**Theorem 2.2** Let $A \in \mathbb{C}^{m \times n}$. For any matrix $V \in \mathbb{C}^{n \times m}$ which satisfies that $\text{rank}(VA) = \text{rank}(V)$, we have

$$(VA)^\dagger V = A_{\mathcal{N}(VA)^\perp,\mathcal{N}(V)}^{(2,4)}.$$ 

**Proof.** The proof follows from Theorem 1.1 and the fact that $X = A_{T,S}^{(2,3)}$ is satisfied if and only if $X^* = A_{S^\perp,T^\perp}^{(2,4)}$. □

**Corollary 1.3** Let $A \in \mathbb{C}^{m \times n}$ and $V \in \mathbb{C}^{n \times m}$ be matrix which satisfies that $\text{rank}(VA) = \text{rank}(V)$ and $P_{\mathcal{N}(VA)^\perp}$ be orthogonal projection on $\mathcal{N}(VA)^\perp$. Then

$$(VA)^\dagger V = P_{\mathcal{N}(VA)^\perp}(VA)^\dagger V = A_{\mathcal{N}(VA)^\perp,\mathcal{N}(V)}^{(2,4)}.$$ 

for arbitrary $(VA)^\dagger \in (VA)\{1\}$.

**Corollary 1.4** Let $A \in \mathbb{C}^{m \times n}$, $T$ and $S$ be subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. For any matrix $V \in \mathbb{C}^{n \times m}$ which satisfies $\text{rank}(VA) = \dim S^\perp$ and $\mathcal{N}(V) = S$, we have $A_{T,S}^{(2,4)} = (VA)^\dagger V = P_{\mathcal{N}(VA)^\perp}(VA)^\dagger V$ if and only if $\mathcal{N}(VA) = T^\perp$, where $P_{\mathcal{N}(VA)^\perp}$ is the orthogonal projection on $\mathcal{N}(VA)^\perp$. Furthermore, $P_{\mathcal{N}(VA)^\perp}(VA)^\dagger V$ is independent of the choice of $(VA)^\dagger \in (VA)\{1\}$.

Remark that Theorem 1.4 is a similar result as Theorems 2.4 and 2.6 from [6]. In [6], authors once again in Theorems 2.3, 2.4, considered arbitrary hermitian matrices $M$ such that $\mathcal{R}(M) = T^\perp$ (it is unique!) and presented one result on few similar ways. Also, we must remark that all the results from [7] follow as analogous results from [6] using the fact that

$$X = A_{T,S}^{(2,3)} \Leftrightarrow X^* = A_{S^\perp,T^\perp}^{(2,4)}.$$
References


