Additive results for the Drazin inverse of block matrices and applications

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Abstract

In this paper we consider the Drazin inverse of a sum of two matrices and we derive additive formulas under conditions weaker than those used in some recent papers on the subject. Like a corollary we get the main results from the paper of H. Yang, X. Liu (The Drazin inverse of the sum of two matrices and its applications, Journ.Comp.Appl.Math., 235 (2011) 1412–1417). As an application we give some new representations for the Drazin inverse of a block matrix.

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1 Introduction

Let \( A \) be a square complex matrix. We denote by \( \mathcal{R}(A), \mathcal{N}(A) \) and \( \text{rank}(A) \), the range, the null space and the rank of matrix \( A \), respectively. In addition, by \( \text{ind}(A) \) we denote the smallest nonnegative integer \( k \) such that \( \text{rank}(A^{k+1}) = \text{rank}(A^k) \), called the index of \( A \). For every matrix \( A \in \mathbb{C}^{n \times n} \), such that \( \text{ind}(A) = k \), there exists an unique matrix \( A^d \in \mathbb{C}^{n \times n} \), which satisfies the relations:

\[
A^{k+1}A^d = A^k, \quad A^dAA^d = A^d, \quad AA^d = A^dA.
\]

The matrix \( A^d \) is called the Drazin inverse of \( A \) (see [9, 10]). The case \( \text{ind}(A) = 0 \) is valid if and only if \( A \) is nonsingular, so \( A^d \) reduces to \( A^{-1} \).
By $A^n = I - AA^d$ we denote the projection on $N(A^k)$ along $R(A^k)$. If the lower limit of a sum is greater than its upper limit, we always define the sum to be 0. For example, the sum $\sum_{i=1}^{k-2} * = 0$, for $k \leq 2$. We agree that $A^0 = I$, for any matrix $A$.

Let $P, Q \in \mathbb{C}^{n \times n}$. The open problem of finding explicit formulas for the Drazin inverse of $P + Q$ in terms of $P$, $Q$, $P^d$, $Q^d$ was posed by Drazin in 1958 [9]. Many authors have considered this problem and have provided formulas for $(P + Q)^d$ under some specific conditions for the matrices $P$ and $Q$. Some of them are listed below:

(i) $PQ = QP = 0$ [9];
(ii) $PQ = 0$ [11];
(iii) $P^2Q = 0$ and $PQ^2 = 0$ [2];
(iv) $PQ^2 = 0$ and $PQP = 0$ [13].

In Section 2 we derive some formulas for $(P + Q)^d$ under weaker conditions than those given in [9, 11, 2, 13].

Formulas for $(P + Q)^d$ can be very useful for deriving formulas for the Drazin inverse of a $2 \times 2$ block matrix. Actually, in 1979 Campbell and Meyer [4] posed the problem of finding an explicit representation for the Drazin inverse of a complex block matrix:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

(1.1)

in terms of its blocks, where $A$ and $D$ are square matrices, not necessarily of the same size. No formula for $M^d$ has yet been offered without any restrictions upon the blocks. Special cases of this problem have been studied, so at present time we have some formulas for $M^d$ under certain conditions on the blocks of $M$. In Section 3 we derive some new formulas for $M^d$. These results are generalizations of some of the results from [7, 8].

First, we will state some auxiliary lemmas.

**Lemma 1.1** [1] Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{n \times m}$. Then $(AB)^d = A((BA)^2)^d B$.

**Lemma 1.2** [12] Let $M_1$ and $M_2$ be matrices of a form

$$M_1 = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix}, \quad M_2 = \begin{bmatrix} B & C \\ 0 & A \end{bmatrix}$$
where $A$ and $B$ are square matrices such that ind$(A) = r$, ind$(B) = s$. Then
\[ \max \{r, s\} \leq \text{ind}(M_i) \leq r + s, \ i = 1, 2, \] and
\[ M_1^d = \begin{bmatrix} A^d & 0 \\ S & B^d \end{bmatrix}, \ M_2^d = \begin{bmatrix} B^d & S \\ 0 & A^d \end{bmatrix}, \]
where
\[ S = (B^d)^2 \left( \sum_{i=0}^{r-1} (B^d)^i C A^i \right) A^\pi + B^\pi \left( \sum_{i=0}^{s-1} B^i C (A^d)^i \right) (A^d)^2 - B^d C A^d. \]

2 The Drazin inverse of a sum of two matrices

Let us define for $j \in \mathbb{N}$, the set $U_j = \{(p_1, q_1, p_2, q_2, ..., p_j, q_j) : \sum_{i=1}^{j} p_i + \sum_{i=1}^{j} q_i = j - 1, \ p_i, q_i \in \{0, 1, ..., j - 1\}, \ i = 1, j\}$.

**Theorem 2.1** Let $P, Q \in \mathbb{C}^{n \times n}$ be such that ind$(P) = r$ and ind$(Q) = s$ and $k \in \mathbb{N}$. If
\[ PQ \prod_{i=1}^{k} (P^\pi Q^\pi) = 0, \] (2.1)
for every $(p_1, q_1, p_2, q_2, ..., p_k, q_k) \in U_k$, then
\[ (P + Q)^d = Y_1 + Y_2 \]
\[ + \sum_{i=1}^{k-1} \left( Y_1 (P^d)^i + (Q^d)^i Y_2 - \sum_{j=1}^{i+1} (Q^d)^j (P^d)^i (P^d)^j \right) PQ (P + Q)^{i-1}, \] (2.2)
where
\[ Y_1 = Q^\pi \left( \sum_{i=0}^{s-1} Q^i (P^d)^i \right) P^d, \ Y_2 = Q^d \left( \sum_{i=0}^{r-1} (Q^d)^i P^i \right) P^\pi. \] (2.3)

**Proof**: We will prove this result using mathematical induction on $k$. For $k = 1$ the theorem is true (see [11]). Now, we will assume that it holds for $k - 1$ and let us prove that it holds for $k$.

Using Lemma 1.1 we have that

\[ \text{...} \]

3
\[(P + Q)^d = \begin{bmatrix} I & Q \end{bmatrix} \left( \left( \begin{bmatrix} P \\ I \end{bmatrix} \begin{bmatrix} I & Q \end{bmatrix} \right)^2 \begin{bmatrix} I \\ P \end{bmatrix} \right) = \begin{bmatrix} I & Q \end{bmatrix} \begin{bmatrix} P^2 + PQ \\ P + Q \\ P^2Q + PQ^2 \end{bmatrix}^d \begin{bmatrix} P \\ I \end{bmatrix}. \]

Denote by
\[
M = \begin{bmatrix} P^2 + PQ & P^2Q + PQ^2 \\ P + Q & PQ + Q^2 \end{bmatrix}, \quad M_1 = \begin{bmatrix} PQ & P^2Q + PQ^2 \\ 0 & PQ \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} P^2 & 0 \\ P + Q & Q^2 \end{bmatrix}.
\]

By computation, we show that for arbitrary \(n \in \mathbb{N}\),
\[
M_1^n = \begin{bmatrix} (PQ)^n & W_n \\ 0 & (PQ)^n \end{bmatrix} \quad \text{and} \quad M_2^n = \begin{bmatrix} P^{2n} & 0 \\ S_n & Q^{2n} \end{bmatrix}, \quad (2.4)
\]
where \(W_n = \sum_{i=0}^{n-1} (PQ)^i P(P + Q)Q(PQ)^{n-1-i}\) and \(S_n = \sum_{i=0}^{n-1} Q^2(P + Q)P_{2(n-1-i)}\).

It is evident that \(M_1^n = 0\), for every \(n \geq \frac{k + 1}{2}\). Also, by straightforward computation we have that
\[
M_1 M_2 \prod_{i=1}^{k-1} (M_1^{p_i} M_2^{q_i}) = 0,
\]
for every \((p_1, p_2, q_2, \ldots, p_{k-1}, q_{k-1}) \in U_{k-1}\). Hence, \(M_1\) and \(M_2\) satisfy the conditions of the theorem for \(k - 1\). By induction hypothesis we get that
\[
(M_1 + M_2)^d = Z_1 + Z_2
\]
\[+ \sum_{i=1}^{k-2} \left( Z_1 (M_1^d)^{i+1} + (M_2^d)^{i+1} Z_2 - \sum_{j=1}^{i+1} (M_2^d)^j (M_1^d)^{i+2-j} \right) M_1 M_2 (M_1 + M_2)^{i-1}, \]
where \(Z_1\) and \(Z_2\) are defined by (2.3), in function of matrices \(M_1\) and \(M_2\).

Since \(M_1^d = 0\) and \(M_1^n = I\), we get that \(Z_1 = 0\) and \(Z_2 = \sum_{i=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor-1} (M_2^d)^{i+1} M_1^i\).
Therefore we get

\[(M_1 + M_2)^d = Z_2 + \sum_{i=1}^{k-2} \left( (M_2^d)^{i+1} Z_2 \right) M_1 M_2 (M_1 + M_2)^{i-1}. \quad (2.5)\]

We have that

\[(M_2^d)^n = \begin{bmatrix} (P^d)^{2n} & 0 \\ A_n & (Q^d)^{2n} \end{bmatrix}, \quad (2.6)\]

where \(A_n = Y_1 (P^d)^{2n} + (Q^d)^{2n} Y_2 - \sum_{i=1}^{2n} (Q^d)^i (P^d)^{2n+1-i}, \) for any \(n \in \mathbb{N}.\)

Also, we get that

\[(M_1 + M_2)^n = \begin{bmatrix} P(P + Q)^{2n-1} & P(P + Q)^{2n-1} Q \\ (P + Q)^{2n-1} & (P + Q)^{2n-1} Q \end{bmatrix}, \quad (2.7)\]

for all \(n \in \mathbb{N}.\) Substituting (2.4), (2.6) and (2.7) into (2.5) it completes the proof. \(\square\)

As corollary of Theorem 2.1 in the case \(k = 1,\) we get the main result from [11].

**Corollary 2.1 [11]** Let \(P, Q \in \mathbb{C}^{n \times n}\) be such that \(\text{ind}(P) = r\) and \(\text{ind}(Q) = s.\) If \(PQ = 0\) then

\[(P + Q)^d = Y_1 + Y_2,\]

where \(Y_1\) and \(Y_2\) are defined by (2.3).

If we consider the case where \(k = 2\) in Theorem 2.1 we obtain as a corollary the main result in [13].

**Corollary 2.2 [13]** Let \(P, Q \in \mathbb{C}^{n \times n}\) be such that \(\text{ind}(P) = r\) and \(\text{ind}(Q) = s.\) If \(PQP = 0\) and \(PQ^2 = 0\) then

\[ (P + Q)^d = Y_1 + Y_2 + \left( Y_1 (P^d)^2 + (Q^d)^2 Y_2 - \sum_{i=1}^{2} (Q^d)^i (P^d)^{3-i} \right) PQ. \]

where \(Y_1\) and \(Y_2\) are defined by (2.3).

When \(k = 3,\) we get the following new result.
Corollary 2.3 Let $P, Q \in \mathbb{C}^{n \times n}$ be such that $\text{ind}(P) = r$ and $\text{ind}(Q) = s$. If $PQP^2 = 0$, $PQ^2P = 0$, $PQ = 0$ and $PQ^3 = 0$ then

$$
(P + Q)^d = Y_1 + Y_2 + \left( Y_1(P^d)^2 + (Q^d)^2Y_2 - \sum_{i=1}^{2} (Q^d)^i(P^d)^{3-i} \right) PQ 
$$

$$
+ \left( Y_1(P^d)^3 + (Q^d)^3Y_2 - \sum_{i=1}^{3} (Q^d)^i(P^d)^{4-i} \right) (PQP + PQ^2),
$$

where $Y_1$ and $Y_2$ are defined by (2.3).

3 Applications

Let $M$ be a matrix of the form (1.1), where $A$ and $D$ are square matrices not necessarily of the same size. Throughout this section we assume that $\text{ind}(A) = r$ and $\text{ind}(D) = s$.

The problem of finding $M^d$ was studied in [7], where the authors gave a representation for $M^d$ under the assumptions $BC = 0$, $BD = 0$ and $DC = 0$. This case was extended to the case when $BC = 0$ and $DC = 0$ (see [6]), and also to a case $BC = 0$, $BDC = 0$ and $BD^2 = 0$ (see [8]). In the next theorem we derive an explicit representation of $M^d$, which is an extension of a case when $BC = 0$ and $BD = 0$.

Using the special case of Theorem 2.1 when $k = 3$, we get the following result.

Theorem 3.1 Let $M$ be given by (1.1). If $BCA = 0$, $BCB = 0$, $ABD = 0$ and $CBD = 0$, then

$$
M^d = \begin{bmatrix}
A^d + B\Sigma_1 + ((A^d)^3 + B\Sigma_4)BC & (A^d)^2B + B(D^d)^2 + B\Sigma_2B + B\Sigma_3BD \\
\Sigma_0 + \Sigma_2BC & D^d + \Sigma_1B + \Sigma_2BD
\end{bmatrix},
$$

where

$$
\Sigma_k = (D^d)^2 \sum_{i=0}^{r-1} (D^d)^{i+k}CA^iA^\pi + D^\pi \sum_{i=0}^{s-1} D^iC(A^d)^{i+k}(A^d)^2 
$$

$$
- \sum_{i=0}^{k} (D^d)^{i+1}C(A^d)^{k-i+1}, k \geq 0.
$$

(3.1)
Proof. If we split matrix \( M \) as
\[
M = \begin{bmatrix}
A & 0 \\
C & D
\end{bmatrix} + \begin{bmatrix}
0 & B \\
0 & 0
\end{bmatrix} := P + Q,
\]
we have that \( Q^2 = 0, \ PQP^2 = 0 \) and \( PQPQ = 0 \). Hence, matrices \( P \) and \( Q \) satisfy the conditions of Corollary 2.3 and we get
\[
M^d = P^d + Q(P^d)^2 + (P^d)^2Q + Q(P^d)^3Q + (P^d)^3QP + Q(P^d)^4QP. \tag{3.2}
\]
Now, by Lemma 1.2, we have that for any \( k \geq 1 \),
\[
(P^d)^k = \begin{bmatrix}
(A^d)^k & 0 \\
\Sigma_{k-1} & (D^d)^k
\end{bmatrix}.
\]
After computing all elements of the sum (3.2), we get that the statement of this theorem is valid. □

Corollary 3.1 [8] If \( M \) is matrix of a form (1.1), such that \( BC = 0 \) and \( BD = 0 \), then
\[
M^d = \begin{bmatrix}
A^d & (A^d)^2B \\
\Sigma_0 & D^d + \Sigma_1B
\end{bmatrix},
\]
where \( \Sigma_k, (k \geq 0) \) is defined by (3.1).

In the next theorem we give an extension of a representation for \( M^d \), which is proved in [5].

Theorem 3.2 If matrix \( M \), defined by (1.1), is such that \( BCA = 0, DCA = 0, CBC = 0 \) and \( CBD = 0 \), then
\[
M^d = \begin{bmatrix}
A^d + Z_1C + Z_2CA & Z_0 + Z_2CB \\
(D^d)^2C + C(A^d)^2 + CZ_2C + CZ_3CA & D^d + C^z_1 + ((D^d)^3 + CZ_3)CB
\end{bmatrix},
\]
where
\[
Z_k = (A^d)^2 \sum_{i=0}^{k-1} (A^d)^{i+k}BD^iD^\pi + A^\pi \sum_{i=0}^{k-1} A^i B(D^d)^{i+k}(D^d)^2
\]
\[- \sum_{i=0}^{k} (A^d)^{i+1} B(D^d)^{k-i+1}, k \geq 0. \tag{3.3}
\]
Proof. Using the splitting of matrix $M$

$$
M = \begin{bmatrix}
A & B \\
0 & D
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
0 & C
\end{bmatrix} := P + Q
$$

we get that matrices $P$ and $Q$ satisfy the conditions of Corollary 2.3. Therefore

$$
M^d = P^d + Q(P^d)^2 + (P^d)^2Q + Q(P^d)^3Q + (P^d)^3QP + Q(P^d)^4QP. \quad (3.4)
$$

Using Lemma 1.2, we get

$$
(P^d)^k = \begin{bmatrix}
(A^d)^k & Y_{k-1} \\
0 & (D^d)^k
\end{bmatrix}, \quad (3.5)
$$

where $k \geq 1$. Substituting (3.5) into (3.4) completes the proof. □

As a corollary of Theorem 3.2, we have the following result.

**Corollary 3.2** [5] Let $M$ be given by (1.1) and let $CA = 0$ and $CB = 0$. Then

$$
M^d = \begin{bmatrix}
A^d + Z_1C & Z_0 \\
(D^d)^2C & D^d
\end{bmatrix},
$$

where $Z_k, (k \geq 0)$ is defined by (3.3).

In [8] authors gave an explicit representation for $M^d$ under conditions $BD^\pi C = 0$, $BDD^d = 0$ and $DD^\pi C = 0$. Here we replace the last condition by the two weaker conditions $DD^\pi CA = 0$ and $DD^\pi CB = 0$.

**Theorem 3.3** Let $M$ be given by (1.1). If $BD^\pi C = 0$, $BDD^d = 0$, $DD^\pi CA = 0$ and $DD^\pi CB = 0$, then

$$
M^d = \begin{bmatrix}
A^d + \sum_{i=0}^{s-1}(A^d)^{i+3}BD^iC & \sum_{i=0}^{s-1}(A^d)^{i+2}BD^i \\
\Phi_0 + \sum_{i=0}^{s-1}\Phi_{i+2}BD^iC & D^d + \sum_{i=0}^{s-1}\Phi_{i+1}BD^i
\end{bmatrix}, \quad (3.6)
$$

where

$$
\Phi_k = \sum_{i=0}^{r-1}(D^d)^{i+k+2}CA^iA^\pi + D^\pi C(A^d)^{k+2} - \sum_{i=0}^{k}(D^d)^{i+k}C(A^d)^{k-i+1}, \quad k \geq 0. \quad (3.7)
$$
Proof. First, notice that from conditions $BD\pi C = 0$, $BDDd = 0$, we have that $BD\pi = B$ and $BC = 0$. If we split matrix $M$ as $M = P + Q$, where

$$P = \begin{bmatrix} A & BD\pi \\ C & D^2Dd \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & DD\pi \end{bmatrix},$$

we have $QP^2 = 0$ and $QPQ = 0$. Also, we have that matrix $Q$ is $s$-nilpotent, and therefore $Q^d = 0$. Applying Corollary 2.2 we get

$$M^d = P^d\sum_{i=0}^{s-1}(P^d)^iQ^i + (P^d)^2\sum_{i=0}^{s-1}(P^d)^iQ^iP - P^d. \quad (3.8)$$

Since $BD\pi C = 0$ and $BD\pi D^2Dd = 0$, matrix $P$ satisfies the conditions of Corollary 3.1 and, after computing, we get

$$(P^d)^i = \begin{bmatrix} (A^d)^i & (A^d)^{i+1}BD\pi \\ \Phi_{i-1} & (D^d)^i + \Phi_iBD\pi \end{bmatrix}, \quad \text{for all } i \geq 1, \quad (3.9)$$

where $\Phi_i$ is defined by (3.7). After substituting (3.9) into (3.8) and computing the sum (3.8), we get (3.6). □

**Theorem 3.4** Let $M$ be given by (1.1). If $BD = 0$, $D\pi CA^2 = 0$, $D\pi CAB = 0$ and $D\pi CBC = 0$, then

$$M^d = \begin{bmatrix} A^d + (A^d)^3BC + (A^d)^4BCA & (A^d)^2B + (A^d)^4BCB \\ \Psi_0 + \Psi_2BC + \Psi_3BCA & D^d + \Psi_1B + \Psi_3BCB \end{bmatrix}, \quad (3.10)$$

where

$$\Psi_k = \sum_{i=0}^{r-1}(D^d)^{i+k+2}CA^iA^\pi - \sum_{i=0}^{k}(D^d)^{i+1}C(A^d)^{k-i+1}, \quad k \geq 0. \quad (3.11)$$

**Proof.** We can split matrix $M$ as $M = P + Q$, where

$$P = \begin{bmatrix} A & BD\pi \\ DD^dC & D \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & D\pi C \end{bmatrix}. \quad (3.12)$$

Since $BD = 0, D\pi CA^2 = 0$, $D\pi CAB = 0$ and $D\pi CBC = 0$, we have $QP^2 = 0$ and $QPQ = 0$. Moreover, $Q^d = 0$. Applying Corollary 2.2, we get

$$M^d = P^d + (P^d)^2Q + (P^d)^3QP. \quad (3.13)$$
Matrix $P$ satisfies the conditions of Corollary 3.1, so we get

$$(P^d)^i = \begin{bmatrix} (A^d)^i & (A^d)^{i+1}B \\ \Psi_{i-1} & (D^d)^i + \Psi_iB \end{bmatrix}, \quad i = 1, 2, 3. \quad (3.14)$$

Substituting (3.14) into (3.13) we obtain (3.10). □

**Remark**

1) If the last condition $D^\pi CBC = 0$ from Theorem 3.4 is replaced with two weaker conditions $D^\pi CBCA = 0$ and $D^\pi CB = 0$, then matrices $Q$ and $P$, defined by (3.12), satisfy the conditions of Corollary 2.3. Therefore, we have the following representation for $M_d$:

$$M_d = \begin{bmatrix} A^d + (A^d)^3BC + (A^d)^4BCA + (A^d)^5BCBC & (A^d)^2B + (A^d)^4BCB \\ \Psi_0 + \Psi_2BC + \Psi_3BCA + \Psi_4BCBC & D^d + \Psi_1B + \Psi_3BCB \end{bmatrix},$$

where, for all $k \geq 0$, $\Psi_k$ is defined by (3.11).

2) If conditions $D^\pi CA^2 = 0$, $D^\pi CAB = 0$ and $D^\pi CBC = 0$ are replaced with stronger conditions $D^\pi CA = 0$ and $D^\pi CB = 0$, we have that $BCA = BD^\pi CA = 0$ and $BCB = BD^\pi CB = 0$. Hence, we get the representation from Theorem 2.7 [8] as a corollary of Theorem 3.4.

## 4 Numerical examples

In this section we give two examples as illustrations of Theorems 3.1 and 3.2. In the following example a $2 \times 2$ block matrix $M$ is given, which does not satisfy the conditions from [6, 7, 8]. The representation for $M_d$ is obtained applying Theorem 3.1.

**Example 4.1** Let $M$ be a matrix of the form (1.1), where

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & -1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 1 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 3 & 0 & 5 \\ 1 & 0 & -1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 3 & 3 & 3 & 3 \\ 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since $BC \neq 0$, representations for $M_d$ from [6, 7, 8] fail to apply. After calculating, we get that $BCA = 0$, $BCB = 0$, $ABD = 0$ and $CBD = 0$. 


Hence, the conditions of Theorem 3.1 are satisfied and after applying it we have that

\[
M^d = \begin{bmatrix}
645 & 1419 & 327 & 645 & 99 & 20 & 99 & 2851 \\
10892 & 5446 & 2723 & 10892 & 10892 & 2723 & 10892 & 5446 \\
-\frac{1}{4} & \frac{1}{2} & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} \\
-1433 & 115 & 645 & -1433 & -2525 & 40 & -2525 & -2979 \\
10892 & 5446 & 2723 & 10892 & 10892 & 2723 & 10892 & 5446 \\
645 & 1419 & 327 & 645 & 99 & 20 & 99 & 2595 \\
10892 & 5446 & 2723 & 10892 & 10892 & 2723 & 10892 & 5446 \\
-209 & -1549 & 236 & -209 & 2045 & 302 & 2045 & 299 \\
10892 & 5446 & 2723 & 10892 & 10892 & 2723 & 10892 & 5446 \\
633 & -1875 & 363 & 633 & 1389 & 297 & 1389 & 267 \\
10892 & 5446 & 2723 & 10892 & 10892 & 2723 & 10892 & 5446 \\
1475 & -2201 & 962 & 1475 & 733 & 292 & 733 & 235 \\
10892 & 5446 & 2723 & 10892 & 10892 & 2723 & 10892 & 5446 \\
-251 & 537 & 544 & -251 & 1609 & 50 & 1609 & 2403 \\
10892 & 5446 & 2723 & 10892 & 10892 & 2723 & 10892 & 5446
\end{bmatrix}
\].

The next example describes a $2 \times 2$ block matrix $M$, for which $M^d$ cannot be derived from the conditions given in [5]. However, we can apply Theorem 3.2 to obtain $M^d$.

**Example 4.2** Let $M$ be a matrix given by (1.1), where

\[
A = \begin{bmatrix}
-1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
1 & 0 & -1 & 1 \\
-1 & 0 & 1 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & -1 & -1 & 1 \\
0 & 0 & 5 & 0 \\
1 & -1 & -1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
1 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 \\
1 & 0 & -1 & -1 \\
1 & 0 & -1 & -1
\end{bmatrix}, \quad D = \begin{bmatrix}
1 & 0 & -1 & -1 \\
0 & 3 & 0 & 5 \\
1 & 0 & -1 & -1 \\
-1 & 0 & 0 & 1
\end{bmatrix}.
\]

We get that $CB \neq 0$, so we can not apply a representation for $M^d$ from [5]. It can be checked that $BCA = 0$, $DCA = 0$, $CBC = 0$ and $CBD = 0$. Therefore we can apply Theorem 3.2 and we get
\[ M^d = \begin{bmatrix}
0 & \frac{5}{68} & 0 & -\frac{25}{136} & -\frac{93}{544} & 0 & \frac{93}{272} & \frac{93}{544} \\
-\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 1 & 0 \\
0 & \frac{5}{68} & 0 & -\frac{25}{136} & \frac{43}{544} & 0 & -\frac{43}{272} & \frac{43}{544} \\
0 & 0 & 0 & \frac{1}{2} & \frac{1}{8} & 0 & \frac{1}{4} & \frac{1}{8} \\
0 & \frac{5}{68} & 0 & -\frac{93}{136} & -\frac{25}{544} & 0 & -\frac{25}{272} & -\frac{25}{544} \\
0 & \frac{3}{34} & 0 & -\frac{15}{68} & -\frac{15}{272} & 0 & \frac{15}{136} & -\frac{15}{272} \\
0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{8} & 0 & -\frac{1}{4} & -\frac{1}{8} \\
0 & \frac{5}{68} & 0 & \frac{43}{136} & -\frac{25}{544} & 0 & -\frac{25}{272} & -\frac{25}{544} 
\end{bmatrix} \]

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References


