A Note on Generalized Inverses and a Block-Rank Equation

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Abstract

In this paper we study the rank equation \( \text{rank} \begin{bmatrix} A & B \\ C & X \end{bmatrix} = \text{rank}(A) \)
and find the necessary and sufficient conditions when \( X = A^{(1,2)} \) and \( X = A^d \) are the solutions of that equation. In both cases we give an explicit form of matrices \( B \) and \( C \).

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1 Introduction

Let \( C^{m \times n} \) denote the set of complex \( m \times n \) matrices. \( I_n \) denotes the unit matrix of order \( n \). By \( A^*, R(A), \text{rank}(A) \) and \( N(A) \) we denote the conjugate transpose, the range, the rank and the null space of \( A \in C^{n \times m} \). The symbol \( A^- \) stands for an arbitrary generalized inner inverse of \( A \), i.e. \( A^- \) satisfies \( AA^-A = A \). By \( A^\dagger \) we denote the Moore-Penrose inverse of \( A \), i.e. the unique matrix \( A^\dagger \) satisfying

\[
AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.
\]

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For $A \in C^{n \times n}$ the smallest nonnegative integer $k$ such that $\text{rank}(A^{k+1}) = \text{rank}(A^k)$ is called the index of $A$ and denoted by $\text{ind}(A)$. If $A \in C^{n \times n}$, with $\text{ind}(A) = k$, then the matrix $X \in C^{n \times n}$ which satisfies the following conditions
$$A^kXA = A^k, \quad XAX = X, \quad AX = XA,$$
is called the Drazin inverse of $A$ and it is denoted by $A^d$. When $\text{ind}(A) = 1$ then the Drazin inverse $A^d$ is called the group inverse and it is denoted by $A^#$. Also, the matrix $X$ which satisfies
$$AXA = A \quad \text{and} \quad XAX = X$$
is called the reflexive inverse of $A$ and it is denoted by $A^{(1,2)}$. For other important properties of generalized inverses see [1] and [3].

In this paper we will consider the rank equation
$$\text{rank} \begin{bmatrix} A & B \\ C & X \end{bmatrix} = \text{rank}(A), \quad (1)$$
for arbitrary $A \in C^{n \times n}$. First, we give a necessary and sufficient conditions such that $X = A^{(1,2)}$ is the solution of equation (1) and all possible matrices $B$ and $C$ are described. As a corollary we obtain the result of J. Gross [8] and N. Thome and Y. Wei [7]. Moreover, we consider when $X = A^d$ is the solution of the equation (1), for an arbitrary matrix $A$ with $\text{ind}(A) = k \geq 1$ and we obtain some interesting corollaries.

## 2 Main results

We start this section with some well-known results. The following lemma was proved in [4], [5] and [6].

**Lemma 2.1** Let $A \in C^{n \times n}, B \in C^{n \times m}, C \in C^{m \times n}$ and $X \in C^{m \times m}$. Then
$$\text{rank} \begin{bmatrix} A & B \\ C & X \end{bmatrix} = \text{rank}(A) + \text{rank}(L) + \text{rank}(M) + \text{rank}(W),$$
where $S = I_n - A^{-1}A$, $L = CS$, $M = SB$ and $W = (I_m - LL^{-})(X - CA^{-}B)(I_m - M^{-}M)$.

The following theorem, which is proved by J. Gross [8], gives a characterization of the existence of the solution (1) by means of geometrical conditions.
**Theorem 2.1** Let $A \in C^{m \times n}, B \in C^{m \times m}$ and $C \in C^{n \times n}$. Then there exists a solution $X \in C^{n \times m}$ of the equation (1) if and only if $R(B) \subseteq R(A)$ and $R(C^\ast) \subseteq R(A^\ast)$, in which case $X = CA^\dagger B$.

Notice that the conditions $R(B) \subseteq R(A)$ and $R(C^\ast) \subseteq R(A^\ast)$ are equivalent to $AA^\dagger B = B$ and $CA^\dagger A = C$. Also, the matrix product $CA^{-}B$ is invariant with respect to the choice of generalized inverse $A^{-}$ of $A$ if and only if $R(B) \subseteq R(A)$ and $R(C^\ast) \subseteq R(A^\ast)$.

First we consider a necessary and sufficient condition such that $X = A^{(1,2)}$ is the solution of the equation (1) and in this case we find the explicit form for $B$ and $C$.

Matrix $A \in C^{m \times n}$ such that $\text{rank}(A) = r$ can be decomposed by

$$A = P \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} Q,$$  \hspace{1cm} \text{(2)}

where $P \in C^{m \times m}, Q \in C^{n \times n}$ and $D \in C^{r \times r}$ are invertible matrices. Given that decomposition arbitrary reflexive generalized inverse of $A$ has the following form

$$A^{(1,2)} = Q^{-1} \begin{bmatrix} D^{-1} & U \\ V & VDU \end{bmatrix} P^{-1},$$ \hspace{1cm} \text{(3)}

where $U$ and $V$ are arbitrary matrices of suitable size (see [2]).

The following theorem gives a sufficient and necessary conditions such that $X = A^{(1,2)}$ is the solution of the equation (1).

**Theorem 2.2** Let $A \in C^{m \times n}, B \in C^{m \times m}$, $C \in C^{n \times n}$ and $X \in C^{n \times m}$ and let the matrix $A$ and its reflexive generalized inverse be given by (2) and (3) respectively. Then $X = A^{(1,2)}$ is the solution of the equation (1) if and only if

$$B = P \begin{bmatrix} DL & (DLD)U \\ 0 & 0 \end{bmatrix} P^{-1} \quad \text{and} \quad C = Q^{-1} \begin{bmatrix} D^{-1}L^{-1} & 0 \\ VL^{-1} & 0 \end{bmatrix} Q,$$ \hspace{1cm} \text{(4)}

for some nonsingular matrix $L \in C^{r \times r}$. 

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Proof. Suppose that $X = A^{(1,2)}$ is the solution of the equation (1). Then there exist matrices $G \in C^{n \times m}$ and $F \in C^{n \times m}$ such that $B = AG$ and $C = FA$ and $CA^{-1}B = A^{(1,2)}$. Let

$$QGP = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \quad \text{and} \quad QFP = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix}.$$ 

Hence,

$$B = AG = P \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} P^{-1} = P \begin{bmatrix} DG_1 & DG_2 \\ 0 & 0 \end{bmatrix} P^{-1} \quad (5)$$

and

$$C = FA = Q^{-1} \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} Q = Q^{-1} \begin{bmatrix} F_1D & 0 \\ F_3D & 0 \end{bmatrix} Q. \quad (6)$$

Also,

$$A^{(1,2)} = FAG = Q^{-1} \begin{bmatrix} F_1DG_1 & F_1DG_2 \\ F_3DG_1 & F_3DG_2 \end{bmatrix} P^{-1}.$$ 

Now, from (3) we have that

$$F_1DG_1 = D^{-1}, \quad F_1DG_2 = U, \quad F_3DG_1 = V.$$ 

From the first equation we obtain that $F_1, G_1$ are invertible matrices and $F_1D = D^{-1}G_1^{-1}$. Now, $DG_2 = F_1^{-1}U = DG_1DU$ and $F_3D = VG_1^{-1}$. If we replace that in (5) and (6) and put $G_1 = L$, we obtain (4).

Now, suppose that (4) holds. Then $AA^{-1}B = B$ and $C = CA^{-1}A$, for generalized inner inverse $A^{-1}$ of $A$, which is given by

$$A^{-1} = Q^{-1} \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$ 

So by Theorem 2.1 there exists a solution $X = CA^{-1}B$ of the equation (1). By (4) we can easily check that $X = CA^{-1}B = A^{(1,2)}$. \hfill \Box

Remark that when we consider the special reflexive inverse of $A$,

$$A^{(1,2)} = Q^{-1} \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}, \quad (7)$$

for $U = V = 0$, we obtain the ([7], Theorem 3).
Corollary 2.1 Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times m}$, $C \in \mathbb{C}^{n \times n}$ and $X \in \mathbb{C}^{n \times m}$. Let the matrix $A$ and one its reflexive generalized inverses be given by (2) and (7) respectively. Then $X = A^{(1,2)}$ is the solution of the equation (1) if and only if

$$B = P \begin{bmatrix} DL & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \quad \text{and} \quad C = Q^{-1} \begin{bmatrix} D^{-1}L^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q,$$

for some nonsingular matrix $L \in \mathbb{C}^{r \times r}$.

Now, we consider the singular value decomposition of $A \in \mathbb{C}^{m \times n}$ such that $\text{rank}(A) = r$

$$A = M \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} N^*,$$

where $M \in \mathbb{C}^{m \times m}$ and $N \in \mathbb{C}^{n \times n}$ are unitary and $D \in \mathbb{C}^{r \times r}$ is a real positive definite diagonal matrix. By Theorem 2.2 we obtain ([8], Theorem 2).

Corollary 2.2 Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times m}$, $C \in \mathbb{C}^{n \times n}$ and $X \in \mathbb{C}^{n \times m}$. Let the matrix $A$ be given by (9). Then $X = A^\dagger$ is the solution of the equation (1) if and only if

$$B = M \begin{bmatrix} DL & 0 \\ 0 & 0 \end{bmatrix} M^* \quad \text{and} \quad C = N \begin{bmatrix} D^{-1}L^{-1} & 0 \\ 0 & 0 \end{bmatrix} N^*,$$

for some nonsingular matrix $L \in \mathbb{C}^{r \times r}$.

Proof. Taking $P = M$ and $Q = N^*$ in (2) we obtain that the matrix $A$ has the representation (9) and in that case $A^\dagger = N \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} M^*$, which has the form (7). Hence, the result follows from Corollary 2.1. □

In the rest of the paper, we consider the following question: When $X = A^d$ is the solution of the equation (1)?

First, let $A \in \mathbb{C}^{m \times n}$ and $\text{ind}(A) = 1$. Using the Jordan canonical form of $A$, there exist nonsingular matrices $P \in \mathbb{C}^{m \times n}$ and $D \in \mathbb{C}^{r \times r}$ such that

$$A = P \begin{bmatrix} D^* & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$ 

We obtain the result of N.Thome and Y. Wei ([7], Theorem 2).
Theorem 2.3 Let \( A \in \mathbb{C}^{n \times n} \) with \( \text{ind}(A) = 1 \) and \( \text{rank}(A) = r \) be given by (11), let \( B, C, X \in \mathbb{C}^{n \times n} \). Then \( X = A^\# \) is the solution of the equation (1) if and only if
\[
B = P \begin{bmatrix} DL & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \quad \text{and} \quad C = P \begin{bmatrix} D^{-1}L^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}, \tag{12}
\]
for some nonsingular matrix \( L \in \mathbb{C}^{r \times r} \).

Proof. If the matrix \( A \) is given by (11), then \( A^\# = P \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \).

Hence, the result follows from Corollary 2.1 taking \( Q = P^{-1} \) and noticing that \( A^\# \) is given by (7).

Now, we consider a more general case when \( A \in \mathbb{C}^{n \times n} \) is such that \( \text{ind}(A) = k \geq 1 \) and \( \text{rank}(A) = r \). Then the matrix \( A \) can be written as
\[
A = P^{-1} \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} P, \tag{13}
\]
where \( P \in \mathbb{C}^{n \times n} \), \( M \in \mathbb{C}^{r \times r} \) are nonsingular matrices and \( N \in \mathbb{C}^{(n-r) \times (n-r)} \) is nilpotent, that is \( N^k = 0 \). In this case
\[
A^d = P^{-1} \begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} P.
\]

Theorem 2.4 Let \( A \in \mathbb{C}^{n \times n} \), with index \( (A) = k \), be represented by (13) and \( B, C, X \in \mathbb{C}^{n \times n} \). Then \( X = A^d \) is the solution of the equation (1) if and only if there exist \( G_1, F_1 \in \mathbb{C}^{r \times r} \), \( G_2, F_2 \in \mathbb{C}^{r \times (n-r)} \), \( G_3, F_3 \in \mathbb{C}^{(n-r) \times r} \) and \( G_4, F_4 \in \mathbb{C}^{(n-r) \times (n-r)} \) such that
\[
B = P^{-1} \begin{bmatrix} MG_1 & MG_2 \\ NG_3 & NG_4 \end{bmatrix} P \quad \text{and} \quad C = P^{-1} \begin{bmatrix} F_1M & F_2N \\ F_3M & F_4N \end{bmatrix} P \tag{14}
\]
and
\[
F_1MG_1 + F_2NG_3 = M^{-1}, \tag{15}
\]
\[
F_1MG_2 + F_2NG_4 = 0,
\]
\[
F_3MG_1 + F_4NG_3 = 0,
\]
\[
F_3MG_2 + F_4NG_4 = 0.
\]
Proof. Suppose that $X = A^d$ is the solution of the equation (1). From Theorem 2.1 we have that $R(B) \subseteq R(A)$ and $R(C^*) \subseteq R(A^*)$, so there exist matrices $G$ and $F$ such that $B = AG$ and $C = FA$ and $A^d = CA^\dagger B$. Let

$$P G P^{-1} = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} \quad \text{and} \quad P F P^{-1} = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix}.$$ 

It follows that

$$B = P^{-1} \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} P = P^{-1} \begin{bmatrix} MG_1 & MG_2 \\ NG_3 & NG_4 \end{bmatrix} P$$

and

$$C = P^{-1} \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} P = P^{-1} \begin{bmatrix} F_1 M & F_2 N \\ F_3 M & F_4 N \end{bmatrix} P.$$

Since the matrix $A$ has the form (13), it follows that

$$A^d = P^{-1} \begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} P \quad \text{and} \quad A^\dagger = P^{-1} \begin{bmatrix} M^{-1} & 0 \\ 0 & N^\dagger \end{bmatrix} P.$$ 

Hence,

$$A^d = P^{-1} \begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} P$$

$$= P^{-1} \begin{bmatrix} F_1 M & F_2 N \\ F_3 M & F_4 N \end{bmatrix} \begin{bmatrix} M^{-1} & 0 \\ 0 & N^\dagger \end{bmatrix} \begin{bmatrix} MG_1 & MG_2 \\ NG_3 & NG_4 \end{bmatrix} P$$

$$= P^{-1} \begin{bmatrix} F_1 MG_1 + F_2 NG_3 & F_1 MG_2 + F_2 NG_4 \\ F_3 MG_1 + F_4 NG_3 & F_3 MG_2 + F_4 NG_4 \end{bmatrix} P.$$ 

We obtain the following system

$$F_1 MG_1 + F_2 NG_3 = M^{-1},$$

$$F_1 MG_2 + F_2 NG_4 = 0,$$

$$F_3 MG_1 + F_4 NG_3 = 0,$$

$$F_3 MG_2 + F_4 NG_4 = 0.$$ 

Conversely, suppose that the matrices $B$ and $C$ satisfied (14). Then we see that $AA^\dagger B = B$ and $C = CA^\dagger A$. From Theorem 2.1 we have that there
exists a solution $X = CA^\dagger B$ of the equation (1). Now, from the system (15) it follows that $CA^\dagger B = A^d$, so $X = A^d$ is the solution of the equation (1). □

Notice that Theorem 2.4 is a generalization of Theorem 2.3.

Now, we state some interesting results.

**Theorem 2.5** Let $A \in C^{n \times n}$, with $\text{ind}(A) = k$ has the form (13), let $p, m, n$ be positive integers and $m, n \geq k$. Then $X = A^d$ is the solution of the equation

$$\text{rank} \begin{bmatrix} A^p & A^n \\ A^m & X \end{bmatrix} = \text{rank}(A^p),$$

if and only if $M^{m+n-p} = M^{-1}$.

**Proof.** Suppose that $X = A^d$ is the solution of the equation (16). Then $A^d = A^m(A^p)-A^n$. Hence,

$$P^{-1} \begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} P =$$

$$P^{-1} \begin{bmatrix} M^m & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M^{-p} & 0 \\ 0 & (N^p)^{-} \end{bmatrix} \begin{bmatrix} M^n & 0 \\ 0 & 0 \end{bmatrix} P =$$

$$P^{-1} \begin{bmatrix} M^{(m+n-p)} & 0 \\ 0 & 0 \end{bmatrix} P,$$

i.e. $M^{m+n-p} = M^{-1}$.

On the contrary, suppose that $M^{m+n-p} = M^{-1}$. First, we show that there exists a solution $X$ of the equation (16), i.e. that $R(A^n) \subseteq R(A^p)$ and $N(A^p) \subseteq N(A^m)$.

If $y \in R(A^n)$, then there exists $x$ such that $y = A^n x$, i.e.

$$y = P^{-1} \begin{bmatrix} M^n z_1 \\ 0 \end{bmatrix}, \text{ where } P x = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$
implying that $y = A^p x'$, where $x' = P^{-1} \begin{bmatrix} M^{(p-n)z_1} \\ 0 \end{bmatrix}$. Hence, $R(A^n) \subseteq R(A^p)$ and analogously $N(A^p) \subseteq N(A^m)$. Using the same computation as in the first part, we obtain that $X = A^d$ is the solution of the equation (16). □

**Remark** Notice that Theorem 2.5 is also valid if we put $f(n)$ and $g(m)$ instead of $n, m$, where $f, g$ are arbitrary positive functions.

**Corollary 2.3** Let $A \in C^{n \times n}$, with $\text{ind}(A) = k$ has the form (13), let $p, m, n$ be positive integers such that $m, n \geq k$ and $m + n = p - 1$. Then $X = A^d$ is the solution of the equation (16).

**Corollary 2.4** Let $A \in C^{n \times n}$, then

$$\text{rank} \begin{bmatrix} A^{(2l+1)} & A^t \\ A^t & A^d \end{bmatrix} = \text{rank} A^{(2l+1)},$$

for arbitrary integer $l \geq \text{ind}(A)$.

**Corollary 2.5** Let $A \in C^{n \times n}$ and $\text{ind}(A) = 1$, then

$$\text{rank} \begin{bmatrix} A^3 & A \\ A & A^\# \end{bmatrix} = \text{rank}(A^3).$$

**References**


