Successive matrix squaring algorithm for computing outer inverses

Predrag S. Stanimirović* and Dragana S. Cvetković-Ilić

University of Niš, Department of Mathematics, Faculty of Science,
P.O. Box 224, Višegradska 33, 18000 Niš, Serbia†
E-mail: pecko@pmf.ni.ac.yu, gagamaka@ptt.yu

Abstract
In this paper we derive a successive matrix squaring (SMS) algorithm to approximate an outer generalized inverse with prescribed range and null space of a given matrix $A \in \mathbb{C}^{m \times n}$. We generalize the results from the papers [3], [16], [18], and obtain an algorithm for computing various classes of outer generalized inverses of $A$. Instead of particular matrices used in these articles, we use an appropriate matrix $R \in \mathbb{C}^{n \times m}$, $s \leq r$. Numerical examples are presented.

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1 Introduction and preliminaries
Let $\mathbb{C}^{m \times n}$ and $\mathbb{C}^{m \times n}_r$ denote the set of all complex $m \times n$ matrices and all complex $m \times n$ matrices of rank $r$, respectively. $I_n$ denotes the unit matrix of order $n$. By $A^*$, $\mathcal{R}(A)$, rank($A$) and $\mathcal{N}(A)$ we denote the conjugate transpose, the range, the rank and the null space of $A \in \mathbb{C}^{m \times n}$. By Re$z$ and Im$z$ we denote a real and imaginary part of a complex number $z$, respectively.

For $A \in \mathbb{C}^{m \times n}$, the set of inner and outer generalized inverses are defined by the following, respectively:

$$A\{1\} = \{X \in \mathbb{C}^{n \times m} \mid AXA = A\}, \quad A\{2\} = \{X \in \mathbb{C}^{n \times m} \mid XAX = X\}.$$

The set of all outer inverses with prescribed rank $s$ is denoted by $A\{2\}_s$, $0 \leq s \leq r = \text{rank}(A)$. The symbols $A^-$ or $A^{(1)}$ stand for an arbitrary generalized
inner inverse of \( A \) and by \( A^{(2)} \) we denote an arbitrary generalized outer inverse of \( A \). Also, the matrix \( X \) which satisfies

\[
AXA = A \quad \text{and} \quad XAX = X
\]

is called the reflexive \( g \)-inverse of \( A \) and it is denoted by \( A^{(1,2)} \). The set of all reflexive \( g \)-inverses is denoted by \( A\{1, 2\} \). Subsequently, the sets of \( \{1, 2, 3\} \) and \( \{1, 2, 4\} \) inverses of \( A \) are defined by

\[
A\{1, 2, 3\} = A\{1, 2\} \cap \{X \mid (AX)^* = AX\},
\]

\[
A\{1, 2, 4\} = A\{1, 2\} \cap \{X \mid (XA)^* =XA\}.
\]

By \( A^\dagger \) we denote the Moore-Penrose inverse of \( A \), i.e. the unique matrix \( A^\dagger \) satisfying

\[
AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A.
\]

For \( A \in \mathbb{C}^{n \times n} \) the smallest nonnegative integer \( k \) such that \( \text{rank}(A^{k+1}) = \text{rank}(A^k) \) is called the index of \( A \) and denoted by \( \text{ind}(A) \). If \( A \in \mathbb{C}^{n \times n} \) is a square matrix with \( \text{ind}(A) = k \), then the matrix \( X \in \mathbb{C}^{n \times n} \) which satisfies the following conditions

\[
A^kXA = A^k, \quad XAX = X, \quad AX =XA,
\]

is called the Drazin inverse of \( A \) and it is denoted by \( A^D \). When \( \text{ind}(A) = 1 \), Drazin inverse \( A^D \) is called the group inverse and it is denoted by \( A^\# \).

Suppose that \( M \) and \( N \) are Hermite positive definite matrices of the order \( m \) and \( n \), respectively. Then there exists the unique matrix \( X \in \mathbb{C}^{n \times m} \) such that

\[
AXA = A, \quad XAX = X, \quad (MAX)^* = MAX, \quad (NXA)^* = NXA.
\]

The matrix \( X \) is called the weighted Moore-Penrose inverse of \( A \), and denoted by \( X = A^\dagger_{M,N} \). In particular, if \( M = I_m \) and \( N = I_n \), then \( A^\dagger_{M,N} = A^\dagger \).

If \( A \in \mathbb{C}^{m \times n} \) and \( W \in \mathbb{C}^{m \times n} \), then the unique solution \( X \in \mathbb{C}^{n \times m} \) of the equations

\[
(WA)^{k+1}XW = (WA)^k, \quad XWAWX = X, \quad AWX = XWA, \quad (1.1)
\]

where \( k = \text{ind}(AW) \), is called the W-weighted Drazin inverse of \( A \) and it is denoted by \( A^{D,W} \).

If \( A \in \mathbb{C}^{m \times n} \), \( T \) is a subspace of \( \mathbb{C}^n \) of dimension \( t \leq r \) and \( S \) is a subspace of \( \mathbb{C}^m \) of dimension \( m - t \), then \( A \) has a \( \{2\} \) inverse \( X \) such that \( R(X) = V \) and \( N(X) = U \) if and only if

\[
AV \oplus U = \mathbb{C}^m,
\]

in which case \( X \) is unique and we denote it by \( A^{(2)}_{V,U} \).
It is well-known that for $A \in \mathbb{C}^{m \times n}$, the Moore-Penrose inverse $A^\dagger_{M,N}$ and the weighted Drazin inverse $A^{D,W}$ can be represented by:

(i) $A^\dagger = A^{(2)}_{R(A^*),N(A^*)}$,

(ii) $A^\dagger_{M,N} = A^{(2)}_{R(A^*),N(A^*)}$, where $A^\dagger = N^{-1}A^*M$,

(iii) $A^{D,W} = (WAW)^{(2)}_{R(A(WA)^*),N(A(WA)^*)}$, where $W \in \mathbb{C}^{n \times n}$, $k = \text{ind}(WA)$.

Also, for $A \in \mathbb{C}^{n \times n}$, the Drazin inverse $A^D$ can be represented by:

$$A^D = A^{(2)}_{R(A^s),N(A^s)}, \quad \text{where } \text{ind}(A) = k.$$ 

The following representations of $\{2, 3\}$, $\{2, 4\}$-inverses with prescribed rank $s$ are restated from [11]:

**Proposition 1.1** Let $A \in \mathbb{C}^{m \times n}$ and $0 < s < r$ be a chosen integer. Then the following is valid:

(a) $A\{2, 4\} = \{ (ZA)^\dagger Z \mid Z \in \mathbb{C}^{s \times m}, ZA \in \mathbb{C}^{s \times n} \}$;

(b) $A\{2, 3\} = \{ (YA)^\dagger Y \mid Y \in \mathbb{C}^{n \times s}, AY \in \mathbb{C}^{n \times s} \}$.

General representations for various classes of generalized inverses can be found in [4, 8, 10, 12]. Some of these representations are restated here for the sake of completeness.

**Proposition 1.2** Let $A \in \mathbb{C}^{m \times n}$ be an arbitrary matrix and $A = PQ$ is a full-rank factorization of $A$. There are the following general representations for some classes of generalized inverses:

$$A\{2\} = \{ F(GAF)^{-1}G \mid F \in \mathbb{C}^{n \times s}, G \in \mathbb{C}^{s \times n}, \text{rank}(GAF) = s \};$$

$$A\{2\} = \bigcup_{s=0}^{r} A\{2\}_s;$$

$$A\{2, 3\} = \{ F(GAF)^{-1}G \mid F \in \mathbb{C}^{n \times s}, G \in \mathbb{C}^{s \times m}, \text{rank}(GAF) = r \} = A\{2\}_r;$$

$$A\{1, 2\} = \{ (P^*AF)^{-1}P^* \mid F \in \mathbb{C}^{n \times r}, \text{rank}(P^*AF) = r \};$$

$$A\{1, 2\} = \{ Q^*(GAQ^*)^{-1}G \mid G \in \mathbb{C}^{r \times m}, \text{rank}(GAQ^*) = r \};$$

$$A^D = P_{A^l}(Q_{A^l} A P_{A^l})^{-1}Q_{A^l}, \quad A^l = P_{A^l}Q_{A^l}, \quad l \geq \text{ind}(A).$$

For other important properties of generalized inverses see [1, 2, 6, 13].

We will use the following well-known result:

**Lemma 1.1** [7] Let $M \in \mathbb{C}^{n \times n}$ and $\varepsilon > 0$ be given. There is at least one matrix norm $\| \cdot \|$ such that

$$\rho(M) \leq \| M \| \leq \rho(M) + \varepsilon,$$

where $\rho(M)$ denotes the spectral radius of $M$. 

The basic motivation for this paper were the paper of L. Chen et al. [3] and the papers of Y. Wei [16] and Y. Wei et al. [18]. In the paper of L. Chen et al. [3] a successive matrix squaring (SMS) algorithm which approximates the Moore-Penrose inverse of a given matrix \(A\) were considering, while in the papers of Y. Wei [16] and Y. Wei et al. [18] the authors considered two variants of SMS algorithm which approximate the Drazin inverse and the weighted Moore-Penrose inverse of \(A\), respectively.

In this paper we derive a SMS algorithm to approximate an outer generalized inverse with prescribed range and null space of a given matrix \(A \in \mathbb{C}^{m \times n}\), which is based on successive squaring of an appropriate block matrix \(T\). We generalized the results from the papers [3], [16] and [18]. Instead of matrices \(A^*, A^k, k \geq \text{ind}(A)\) and \(A^\#\), used to compose the matrix \(T\) in these articles, we use an universal appropriate matrix \(R \in \mathbb{C}^{n \times m}, 0 \leq s \leq r\). Furthermore, we give an explicit form of the outer generalized inverse of \(A\) corresponding to chosen \(R\). An explicit approximation of rank(\(X\)) is also given in the same iterative scheme. We also estimate the number of iterative steps necessary to obtain a prescribed precision. Numerical examples in the last section show that numerical approximation of rank(\(X\)) is important and that the number of SMS iterations is less than in the classical Euler-Knopp iterations.

2 Results

Let \(A \in \mathbb{C}^{m \times n}\) and \(R \in \mathbb{C}^{n \times m}, 0 \leq s \leq r\) be given. The general iterative scheme used in this paper is given by

\[
X_1 = Q, \\
X_{k+1} = PX_k + Q, \ k \in \mathbb{N},
\]

(2.1)

where \(P = I - \beta RA\), \(Q = \beta R\) and \(\beta\) is a relaxation parameter.

The iterative process (2.1) can be accelerated by means of \((m+n) \times (m+n)\) composite matrix \(T\), partitioned in the block form

\[
T = \begin{bmatrix}
P & Q \\
0 & I
\end{bmatrix}.
\]

(2.2)

The improvement of (2.1) can be done by computing the matrix power

\[
T^k = \begin{bmatrix}
P^k & \sum_{i=0}^{k-1} P^i Q \\
0 & I
\end{bmatrix}.
\]

It is not difficult to see that the iterative scheme (2.1) gives \(X_k = \sum_{i=0}^{k-1} P^i Q\). Hence, the matrix \(X_k\) is equal to the right upper block in \(T^k\). In turn, \(T^{2k}\) can be computed by \(k\) repeated squaring, that is

\[
T_0 = T, \\
T_{i+1} = T_i^2, \ i = 0, 1, \ldots, k - 1.
\]

(2.3)
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It is clear that

\[ T_k = T^{2^k} = \begin{bmatrix} P^{2^k} & \sum_{i=0}^{2^k-1} P^i Q \\ 0 & I \end{bmatrix}. \]  

(2.4)

Now, we will state an auxiliary result:

**Lemma 2.1** If \( P \in \mathbb{C}^{n \times n} \) and \( S \in \mathbb{C}^{n \times n} \) are such that \( P = P^2 \) and \( PS = SP \) then

\[ \rho(PS) \leq \rho(S). \]

**Proof.** Suppose that \( \lambda \) is an eigenvalue of \( PS \) and \( x \) is a corresponding vector such that \( PSx = \lambda x \). Then, \( \lambda P x = \lambda x \) which implies that \( \lambda = 0 \) or \( P x = x \). If \( P x = x \), then \( S x = SP x = PS x = \lambda x \), i.e. \( \lambda \) is an eigenvalue of \( S \). Hence, the set of eigenvalues of \( PS \) is the subset of the set of eigenvalues of \( S \) union \( \{0\} \), which implies that \( \rho(PS) \leq \rho(S) \).

The following theorem represents the analogue of Theorem 3.1 [15] which deals with V-norm, where V-norm is defined as \( \|B\|_V = \|BV\|_2 \), for any \( B \in \mathbb{C}^{n \times m} \) and \( V \) is invertible such that \( V^{-1}AGV \) is the Jordan form of \( AG \).

The method used in our proof is different then the one used in Theorem 3.1 [15].

**Theorem 2.1** Let \( A \in \mathbb{C}^{m \times n} \) and \( R \in \mathbb{C}^{n \times m}, \ 0 \leq s \leq r \) be given such that

\[ AR(R) \oplus N(R) = \mathbb{C}^m. \]  

(2.5)

The sequence of approximations

\[ X_{2^k} = \sum_{i=0}^{2^k-1} (I - \beta RA)^i \beta R \]  

(2.6)

determined by the SMS algorithm (2.3) converges in the matrix norm \( \| \cdot \| \) to the outer inverse \( X = A_{\mathcal{R}(R),N(R)}^{(2)} \) of \( A \) if \( \beta \) is a fixed real number such that

\[ \max_{1 \leq i \leq t} |1 - \beta \lambda_i| < 1, \]  

(2.7)

where \( \text{rank}(RA) = t, \lambda_i, \ i = 1, \ldots, t \) are eigenvalues of \( RA \) and \( \| \cdot \| \) satisfies condition (1.2) from Lemma 1.1 for \( M = I - \beta AR \).

In the case of convergence we have the following error estimate

\[ \frac{\|X - X_{2^k}\|}{\|X\|} \leq \max_{1 \leq i \leq t} |1 - \beta \lambda_i|^{2^k} + O(\varepsilon), \ k \geq 0. \]  

(2.8)

**Proof.** From the condition (2.5), it follows the existence of the outer inverse \( X = A_{\mathcal{R}(R),N(R)}^{(2)} \) of \( A \). It is evident that \( XAR = R \) and \( R = RAX \). Since \( \mathcal{R}(X_{2^k}) \subseteq \mathcal{R}(R) \), we have that

\[ XAX_{2^k} = X_{2^k}. \]
So, we obtain
\[
\|X - X_{2^k}\| = \|XAX - XAX_{2^k}\|
\]
\[
= \|X(AX - AX_{2^k})\|
\]
\[
\leq \|X\| \cdot \|AX - AX_{2^k}\|. \tag{2.9}
\]

Now, let us prove that
\[
AX - AX_k = (AX - AX_1)^k. \tag{2.10}
\]

We will use the mathematical induction.

For \(k = 1\), the equality (2.10) is true. Suppose that (2.10) holds for \(k = p - 1\) and prove that it holds for \(k = p\). Using \(P = I - QA\), we have
\[
AX - AX_p = AX - A(PX_{p-1} + Q)
\]
\[
= AX - A(I - QA)X_{p-1} - AQ
\]
\[
= AX - AX_{p-1} + AQAX_{p-1} - AQ.
\]

Since \(R = RAX\) implies \(Q = QAX\), we get
\[
AX - AX_p = AXAX - AXAX_{p-1} + AQAX_{p-1} - AQAX
\]
\[
= (AX - AQ)(AX - AX_{p-1})
\]
\[
= (AX - AQ)^p = (AX - AX_1)^p.
\]

Now, (2.9) and (2.10) imply
\[
\|X - X_{2^k}\| \leq \|X\| \cdot \|AX - AX_{2^k}\| \leq \|X\| \cdot \|AX - AX_1\|^{2^k}.
\]

If (2.7) is satisfied, we can choose \(\epsilon\) such that \(\max_{1 \leq i \leq t} |1 - \beta \lambda_i| + \epsilon < 1\). For such \(\epsilon\) and \(M = I - \beta AR\), there exists a matrix norm \(\|\cdot\|\) satisfies (1.2). Now,
\[
\frac{\|X - X_{2^k}\|}{\|X\|} \leq \|AX - AX_1\|^{2^k} \leq \left(\max_{1 \leq i \leq t} |1 - \beta \lambda_i| + \epsilon\right)^{2^k}.
\]

The last inequality follows if we apply Lemma 2.1 for \(P = AX\) and \(S = I - AX_1 = I - \beta AR\) and by Lemma 1.1.

Remark that a similar problem as in Theorem 3.1 [15] was discussed in [17] but in a more general case: If \(\{S_n(x)\}\) is a family of continuous real valued functions on open set \(\Omega\) such that \(\sigma((RA)|_T) \subset \Omega \subset (0, \infty)\) and \(\lim_{n \to \infty} S_n(x) = 1/x\) uniformly on \(\sigma((RA)|_T)\), then
\[
A_{T,S}^{(2)} = \lim_{n \to \infty} S_n((RA)|_T)R
\]

and
\[
\frac{\|S_n((RA)|_T)R - A_{T,S}^{(2)}\|_V}{\|A_{T,S}^{(2)}\|_V} \leq \max_{x \in \sigma((RA)|_T)} |S_n(x)x - 1| + O(\epsilon), \quad \epsilon > 0.
\]

The generalization to the bounded linear operator on Banach space of the mentioned result proved in [17] is given in [5].
Corollary 2.1 Under the assumption and the conditions of Theorem 2.1, 

(i) In the case \( m = n, R = A^l, l \geq \text{ind}(A) \), we have \( A^D = \lim_{k \to \infty} X_{2^k} \).

(ii) In the case \( R = A^*, A^\dagger = \lim_{k \to \infty} X_{2^k} \).

(iii) In the case \( R = A^2 = N^{-1}A^*M, A^\dagger_{M,N} = \lim_{k \to \infty} X_{2^k} \).

(iv) In the case \( R = A(WA)^k, A^D,W = \lim_{k \to \infty} X_{2^k} \).

Corollary 2.2 In the case \( \text{rank}(R) = \text{rank}(A) \), under the conditions of Theorem 2.1, we have \( X \in A \{ 1, 2 \} \).

Proof. Using that \( \text{rank}(X) = \text{rank}(A) \), by Corollary 1 (pg. 19, [1]) it follows that \( X \in A \{ 1, 2 \} \). □

Theorem 2.2 Let \( A \in \mathbb{R}^{m \times n} \) and \( R \in \mathbb{R}^{n \times m} \), \( 0 \leq s \leq r \) be real matrices and assume that \( \beta \) satisfies condition (2.7). Then the sequence of real numbers defined by

\[
r_{2^k} = n - \text{Tr}((I - \beta RA)^{2^k}) = n - \text{Tr}(P^{2^k}), \quad k \geq 0 \tag{2.11}
\]

converges to \( \text{rank}(RA) \).

Proof. It is sufficient to use

\[
\text{Tr}((I - \beta RA)^{2^k}) = n - t + \sum_{i=1}^{t} (1 - \beta \lambda_i)^{2^k}, \tag{2.12}
\]

where \( t = \text{rank}(RA) \leq \text{rank}(R) \). Therefore, using (2.12), we get

\[
r_{2^k} = t - \sum_{i=1}^{t} (1 - \beta \lambda_i)^{2^k}. \tag{2.13}
\]

Now, the proof can be completed using (2.7). □

Remark 2.1 Taking \( R = A^* \) in Theorem 2.1 and Theorem 2.2 we get the results of Theorem 2 from [3]. Also, taking \( R = A^l, l \geq \text{ind}(A) \) and \( R = A^2 \) in Theorem 2.1, respectively, we get the results from [16] and [18].

The next theorem presents an explicit representation of the outer inverse of \( A \) which is a limit value of the sequence of approximations (2.1), for given \( A \) and \( R \).
Theorem 2.3 Assume that $A \in \mathbb{C}^{m \times n}$. Consider an arbitrary matrix $R \in \mathbb{C}^{n \times m}$, $0 \leq s \leq r$, and its full rank factorization $R = FG$, $F \in \mathbb{C}^{n \times s}$, $G \in \mathbb{C}^{s \times m}$, such that $GAF$ is invertible. The sequence of approximations

$$X_{2k} = \sum_{i=0}^{2^k-1} (I - \beta RA)^i \beta R$$  \hspace{1cm} (2.14)

determined by the SMS algorithm (2.3) converges in the matrix norm to the outer inverse $X = F(GAF)^{-1}G$ of $A$, if $\beta$ is a fixed real number satisfying

$$\max_{1 \leq i \leq s} |1 - \beta \lambda_i| < 1,$$  \hspace{1cm} (2.15)

where $\lambda_i$, $i = 1, \ldots, s$ are eigenvalues for $FGA$. In the case of convergence we have the error estimate

$$\frac{\|X - X_{2k}\|}{\|X\|} \leq \max_{1 \leq i \leq s} |1 - \beta \lambda_i|^{2^k} + O(\varepsilon), \hspace{1cm} k \geq 0,$$  \hspace{1cm} (2.16)

where the matrix norm $\|\cdot\|$ satisfies condition (1.2), for $M = I - \beta AR$.

Proof. Since the matrices $F$ and $G$ satisfy $\text{rank}(GAF) = \text{rank}(F) = \text{rank}(G)$, according to Theorem 1.3.8 (pg. 33, [13]) and [14], we conclude that $X = F(GAF)^{-1}G$ is a \{2\} inverse of $A$ having the range $\mathcal{R}(X) = \mathcal{R}(F) = V$ and null space $\mathcal{N}(X) = \mathcal{N}(G) = U$, i.e.

$$X = A^{(2)}_{V,U} = A^{(2)}_{\mathcal{R}(F), \mathcal{N}(G)}.$$  \hspace{1cm} (2.17)

Also, conditions conditions $AV \cap U = \{0\}$ and $AV \oplus U = \mathbb{C}^m$ for the existence of $X$ are ensured (Theorem 1.3.8, pg. 33, [13]). On the other hand, since

$$\text{rank}(R) = s = \text{rank}(FG) = \text{rank}(F) = \text{rank}(G)$$

we obtain $\mathcal{R}(F) = \mathcal{R}(R), \mathcal{N}(G) = \mathcal{N}(R)$ which implies $X = A^{(2)}_{\mathcal{R}(R), \mathcal{N}(R)}$. In view of (2.15) immediately follows that the conditions of Theorem 2.1 are satisfied. Therefore, we may conclude that the sequence of approximations (2.14) determined by the SMS algorithm (2.3) converges to $X = F(GAF)^{-1}G$, in the matrix norm satisfying (1.2) for $M = I - \beta AR$. \hfill \Box

Corollary 2.3 Assume that $A \in \mathbb{R}^{m \times n}$ and $R \in \mathbb{R}^{n \times m}$, $0 \leq s \leq r$, are two arbitrary chosen real matrices. Let $R = FG$, $F \in \mathbb{R}^{n \times s}$, $G \in \mathbb{R}^{s \times m}$ be a full rank factorization of $R$, such that $GAF$ is invertible. The sequence of real numbers defined by

$$r_{2k} = n - \text{Tr}((I - \beta RA)^{2^k}) = n - \text{Tr}(P^{2^k}), \hspace{1cm} k \geq 0$$  \hspace{1cm} (2.17)

converges to $\text{rank}(X)$.
Proof. According to Theorem 2.2, \( r_{2k} \to \text{rank}(RA) \). Since
\[
\text{rank}(RA) = \text{rank}(GAF) = s = \text{rank}(R) = \text{rank}(X),
\]
we complete the proof.

Corollary 2.4 Consider \( A \in \mathbb{C}^{n \times n} \) and arbitrary chosen matrix \( R \in \mathbb{C}^{n \times m} \), \( s \leq r \). Assume that \( R = FG, F \in \mathbb{C}^{n \times s}, G \in \mathbb{C}^{s \times m} \) is a full-rank factorization of \( R \). Also, suppose that condition (2.15) of Theorem 2.3 are valid and that \( A = PQ \) is a full rank factorization of \( A \). If the sequences of approximations \( X_{2k} \) is defined by (2.14), the following hold:

a) \( \left\{ \lim_{k \to \infty} X_{2k} \mid F \in \mathbb{C}^{2 \times s}, G \in \mathbb{C}^{s \times m}, \text{rank}(GAF) = s, 0 \leq s \leq r \right\} \subseteq A\{2\}; \)

b) \( \left\{ \lim_{k \to \infty} X_{2k} \mid F \in \mathbb{C}^{n \times r}, G \in \mathbb{C}^{r \times m}, \text{rank}(GAF) = r \right\} \subseteq A\{1, 2\}; \)

c) \( \left\{ \lim_{k \to \infty} X_{2k} \mid F \in \mathbb{C}^{n \times r}, G = P^*, \text{rank}(GAF) = r \right\} \subseteq A\{1, 2, 3\}; \)

d) \( \left\{ \lim_{k \to \infty} X_{2k} \mid F = Q^*, G \in \mathbb{C}^{r \times m}, \text{rank}(GAF) = r \right\} \subseteq A\{1, 2, 4\}; \)

Proof. a) If the matrices \( F \in \mathbb{C}^{n \times s} \) and \( G \in \mathbb{C}^{s \times m} \) satisfy \( \text{rank}(GAF) = s \), applying Theorem 2.3, we conclude that \( X = F(GAF)^{-1}G \) is the limit value of the sequence of approximations (2.14). Using general representation of outer inverses form Proposition 1.2, we immediately obtain \( X \in A\{2\} \).

Parts b)-d) can be proved using Theorem 2.3 and general representations of pseudoinverses of \( A \).

Corollary 2.5 Consider matrix \( A \in \mathbb{C}^{m \times n} \) and integer \( s \) such that \( 0 \leq s \leq r \). Then the following is valid:

a) \( \left\{ Y \cdot \lim_{k \to \infty} \left( \sum_{i=0}^{2^{k-1}} (I - \beta(AY)^*AY)^i(AY)^* \right) \mid Y \in \mathbb{C}^{n \times s}, \text{rank}(AY) = s \right\} \subseteq A\{2, 3\}_s; \)

b) \( \left\{ \lim_{k \to \infty} \left( \sum_{i=0}^{2^{k-1}} (I - \beta(ZA)^*ZA)^i(ZA)^* \right) \cdot Z \mid Z \in \mathbb{C}^{s \times n}, \text{rank}(ZA) = s \right\} \subseteq A\{2, 4\}_s. \)

Proof. Follows from Corollary 2.1, part (ii) and the general representations of the sets \( A\{2, 3\}_s \) and \( A\{2, 4\}_s \) given in Proposition 1.1.

In the following theorem we investigate the limits of the relaxation parameter \( \beta \) by which the convergence of the sequence \( X_{2k} \) is ensured. Eigenvalues of \( RA \) are complex numbers, in general.
Theorem 2.4 Let $A \in \mathbb{C}^{m \times n}$ and $R \in \mathbb{C}^{n \times m}$, $0 \leq s \leq r$ be given. Assume that \(\text{rank}(RA) = t\) and \(\lambda_i, \ i = 1, \ldots, t\) are eigenvalues of $RA$. Then the condition
\[
\max_{1 \leq i \leq t} |1 - \beta \lambda_i| < 1
\tag{2.18}
\]
is equivalent with the following:

a) For each $i \in \{1, \ldots, t\}$, $\text{Re} \lambda_i > 0$, and the relaxation parameter $\beta$ satisfies
\[
0 < \beta < 2 \min \left\{ \frac{\text{Re} \lambda_i}{|\lambda_i|^2} \mid i = 1, \ldots, t, \ |\lambda_i| \neq 0 \right\},
\tag{2.19}
\]
or

b) For each $i \in \{1, \ldots, t\}$, $\text{Re} \lambda_i < 0$, and the relaxation parameter $\beta$ satisfies
\[
2 \max \left\{ \frac{\text{Re} \lambda_i}{|\lambda_i|^2} \mid i = 1, \ldots, t, \ |\lambda_i| \neq 0 \right\} < \beta < 0.
\tag{2.20}
\]

Proof. According to (2.18), for each $i \in \{1, \ldots, t\}$ we get
\[
|1 - \beta \lambda_i| = \sqrt{(1 - \beta \text{Re} \lambda_i)^2 + (\beta \text{Im} \lambda_i)^2} < 1.
\]
This implies
\[
\beta(|\lambda_i|^2 - 2 \text{Re} \lambda_i) < 0.
\tag{2.21}
\]

There are two possible cases:

a) If $\beta > 0$, then $\beta|\lambda_i|^2 - 2 \text{Re} \lambda_i < 0$, so $\text{Re} \lambda_i > 0$, for every $i \in \{1, \ldots, t\}$ and $\beta < 2 \frac{\text{Re} \lambda_i}{|\lambda_i|^2}$, for every $\lambda_i$ such that $|\lambda_i| \neq 0$.

b) If $\beta < 0$, then $\beta|\lambda_i|^2 - 2 \text{Re} \lambda_i > 0$, so $\text{Re} \lambda_i < 0$, for every $i \in \{1, \ldots, t\}$
and $\beta > 2 \frac{\text{Re} \lambda_i}{|\lambda_i|^2}$, for every $\lambda_i$ such that $|\lambda_i| \neq 0$. \qed

Now we are looking for practical values of the parameter $\beta$. Let us denote by $m \text{Re} = \min \{\text{Re} \lambda_1, \ldots, \text{Re} \lambda_t\}$, $M \text{Re} = \max \{\text{Re} \lambda_1, \ldots, \text{Re} \lambda_t\}$ and $(M \text{Im})^2 = \max \{(\text{Im} \lambda_1)^2, \ldots, (\text{Im} \lambda_t)^2\}$.

In accordance with conditions a) of Theorem 2.4 and condition (2.19), in the case $\text{Re} \lambda_i > 0$, for every $i \in \{1, \ldots, t\}$, we can use the following value for $\beta$:
\[
\beta^C_{\text{opt}} = \frac{m \text{Re}}{(M \text{Re})^2 + (M \text{Im})^2}.
\tag{2.22}
\]

Similarly, according to conditions b) of Theorem 2.4 and condition (2.20), in the case $\text{Re} \lambda_i < 0$, for every $i \in \{1, \ldots, t\}$, we can use:
\[
\beta^C_{\text{opt}} = \frac{M \text{Re}}{(m \text{Re})^2 + (M \text{Im})^2}.
\tag{2.23}
Theorem 2.5 Let $A \in \mathbb{C}^{m \times n}$ and $R \in \mathbb{C}^{n \times m}$, $0 \leq s \leq r$ be given. Also, suppose that $\text{rank}(RA) = t$ and $\lambda_i$, $i = 1, \ldots, t$ are eigenvalues of $RA$.

a) If $\text{Re} \lambda_i > 0$, for every $i \in \{1, \ldots, t\}$, the following statements are valid:

(i) For $\beta^C_{\text{opt}}$ defined in (2.22) the iterative sequence (2.1) converges.

(ii) The following error estimate is valid:

$$\frac{\|X - X_{2^k}\|}{\|X\|} \leq (1 - \beta^C_{\text{opt}} \cdot \text{Re} \lambda_i)^{2^{k-1}} + O(\varepsilon).$$

(iii) The number $k$ of required iterations needful to attain the accuracy

$$\frac{\|X - X_{2^k}\|}{\|X\|} \leq \delta, \quad 0 < \delta < 1$$

is estimated by

$$k \geq \left\lceil 2 - \frac{F((1 - \beta^C_{\text{opt}} \cdot \text{Re} \lambda_i)^2) + F(\delta)}{1 - \beta^C_{\text{opt}} \cdot \text{Re} \lambda_i} \right\rceil, \quad 1 - \beta^C_{\text{opt}} \cdot \text{Re} \lambda_i \neq 0,$$

where $[x]$ gives the smallest integer greater than or equal to $x$ and

$$F(x) = \log_2 (-\ln(x)).$$

b) In the case $\text{Re} \lambda_i < 0$, for every $i \in \{1, \ldots, t\}$, we obtain analogous results:

(i) For $\beta^C_{\text{opt}}$ defined in (2.23), the iterative sequence (2.1) converges.

(ii) The following error estimate is valid:

$$\frac{\|X - X_{2^k}\|}{\|X\|} \leq (1 - \beta^C_{\text{opt}} \cdot \text{MRe} \lambda_i)^{2^{k-1}} + O(\varepsilon).$$

(iii) The number of required iterations $k$ needful to achieve the accuracy defined in (2.24) is an integer satisfying

$$k \geq \left\lceil 2 - \frac{F((1 - \beta^C_{\text{opt}} \cdot \text{MRe} \lambda_i)^2) + F(\delta)}{1 - \beta^C_{\text{opt}} \cdot \text{MRe} \lambda_i} \right\rceil, \quad 1 - \beta^C_{\text{opt}} \cdot \text{MRe} \lambda_i \neq 0.$$

Proof. We prove part a) of the theorem. For the sake of simplicity, we use the notation $z = \text{MRe} + i \cdot \text{MIm}$.

(i) It is evident that $\beta^C_{\text{opt}}$ satisfies the condition (2.19) from part (a) of Theorem 2.4, so by the Theorem 2.1, we have that the iterative sequence defined by (2.1) converges.

(ii) One can verify the following:

$$|1 - \beta^C_{\text{opt}} \lambda_i| = \sqrt{1 - 2 \frac{\text{MRe} \lambda_i}{|z|^2} \text{Re} \lambda_i + \frac{(\text{MRe})^2}{|z|^4} |\lambda_i|^2}. $$
Therefore

\[
\max |1 - \beta_{\text{opt}}^C \lambda_i| \leq \sqrt{1 - 2 \frac{mRe}{|z|^2} mRe + \frac{(mRe)^2}{|z|^4} |z|^2}
\]

\[
= \sqrt{1 - \beta_{\text{opt}}^C \cdot mRe}
\]

Hence, the error estimate follows from (2.8) in Theorem 2.1.

(iii) The number of iterative steps can be derived from

\[
(1 - \beta_{\text{opt}}^C \cdot mRe)^{2^k-1} = \left((1 - \beta_{\text{opt}}^C \cdot mRe)^2\right)^{2^k-2} \leq \delta.
\]

Part b) can be proved in a similar way. \(\square\)

Remark 2.2

(1) In the both cases, when \(\text{Re} \lambda_i > 0\), for each \(i \in \{1, \ldots, t\}\) or \(\text{Re} \lambda_i < 0\), for each \(i \in \{1, \ldots, t\}\), using \(\beta_{\text{opt}}^C\) we get an exact outer inverse \(X\) of \(A\) in the first iteration if \(\text{Im} \lambda_i = 0\), for every \(i \in \{1, \ldots, t\}\) and \(\text{Re} \lambda_i > 0\), for each \(i \in \{1, \ldots, t\}\) and \(1 - \beta_{\text{opt}}^C \cdot MRe = 0\) the sufficient number of iterative steps is \(k = 1\).

(2) Because of the accumulation of round off errors in the iterative steps of SMS method, we recommend to use the minimal number of steps from (2.25) or (2.27). Therefore, in the case \(\text{Re} \lambda_i > 0, i = 1, \ldots, t\) and \(1 - \beta_{\text{opt}}^C \cdot mRe = 0\) or \(\text{Re} \lambda_i < 0, i = 1, \ldots, t\) and \(1 - \beta_{\text{opt}}^C \cdot MRe = 0\) we have

\[
k = \begin{cases} 
2 - F((1 - \beta_{\text{opt}}^C \cdot mRe)^2) + F(\delta), & \text{Re} \lambda_i > 0, i = 1, \ldots, t \\
2 - F((1 - \beta_{\text{opt}}^C \cdot MRe)^2) + F(\delta), & \text{Re} \lambda_i < 0, i = 1, \ldots, t.
\end{cases}
\]

\[ (2.28) \]

3 Numerical examples

The implementation of the SMS method described in Theorem 2.1 is written in the programming package MATHEMATICA. About the package see, for example [20]. In the next examples we use \(\beta = \beta_{\text{opt}}^C\) defined in (2.22) or (2.23).

Example 3.1 In this example we get exact outer inverse \(X\) of \(A\) in the first iteration as we note in the part (1) of Remark 2.2.

Consider the following \(6 \times 5\) matrix of rank 4, investigated in [9]:

\[
A = \begin{bmatrix}
1 & 2 & 3 & 4 & 1 \\
1 & 3 & 4 & 6 & 2 \\
2 & 3 & 4 & 5 & 3 \\
3 & 4 & 5 & 6 & 4 \\
4 & 5 & 6 & 7 & 6 \\
6 & 6 & 7 & 7 & 8
\end{bmatrix}.
\]

In accordance with Theorem 2.3, we can choose appropriate matrices \(F\) and \(G\) in order to generate \(\{2\}\)-inverses \(F(GAF)^{-1}G\) of \(A\). The matrices \(F\) and
Successive matrix squaring algorithm for computing outer inverses

$G$ must satisfy $\text{rank}(FG) = s \leq 4$, and dimensions of the matrix $FG$ must be $6 \times 5$.

For example, one can choose $F$ and $G$ using the following pattern:

$$F = \begin{bmatrix} F^s_s & 0 \\ 0 & I_s \end{bmatrix}, \quad G = \begin{bmatrix} I_s & 0 \end{bmatrix},$$

where $F^s_s A_s = I_s$ and $A_s$ is the principal minor which contains first $s$ rows and first $s$ columns of $A$. Let us choose $s = 2$. In this case the block $F_2$ is equal to

$$F_2 = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}.$$

Applying (2.22), we get $\beta_{\text{opt}}^c = 1$. Both nonzero eigenvalues of $RA$ are equal to 1, so in this case is $m \text{Re} = M \text{Re} = 1$ and $\text{Im} \lambda_1 = \text{Im} \lambda_2 = 0$. As it is claimed in Remark 2.2, part (1), the outer inverse $X_1$ is obtained by SMS algorithm (2.14) immediately, in the first iteration. Corresponding \{2\}-inverse $X_1 = F(GAF)^{-1}G = FG$ of rank 2 is equal to

$$X_1 = X_{21} = \begin{bmatrix} 3 & -2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example 3.2 Consider the matrix $A$ from Example 3.1, and choose the following matrices $F \in \mathbb{C}^{5 \times 2}$ and $G \in \mathbb{C}^{2 \times 6}$ from [9]:

$$F = \begin{bmatrix} 0 & 0 \\ 2 & 1 \\ 5 & 3 \\ 1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

(3.1)

Since $\text{rank}(FG) = 2$ corresponding \{2\}-inverse $X_1 = F(GAF)^{-1}G$ for $A$ of rank 2 is equal to

$$A^{(2)} = X_1 = \frac{1}{174} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -21 & 19 & -21 & 19 & -21 & 19 \\ 60 & -46 & 60 & -46 & 60 & -46 \\ 39 & -27 & 39 & -27 & 39 & -27 \\ -102 & 84 & -102 & 84 & -102 & 84 \end{bmatrix}.$$

Let us choose the precision $\delta = 10^{-6}$. According to heuristic (2.22) the parameter $\beta$ is equal to $\beta_{\text{opt}}^c = 9.208888 \times 10^{-6}$. In accordance with (2.28), the minimal number of iterative steps is equal to $k = 23$. An application of iterations (2.14) produces the following approximation of $A^{(2)} = X_1$ after 23 iterations:

$$X_{23} = \begin{bmatrix} -0.12069 & 0.109195 & -0.12069 & -0.109195 & -0.12069 & 0.109195 \\ 0.344828 & -0.264368 & 0.344828 & -0.264368 & 0.344828 & -0.264368 \\ 0.224138 & -0.155172 & 0.224138 & -0.155172 & 0.224138 & -0.155172 \\ -0.586207 & 0.482759 & -0.586207 & 0.482759 & -0.586207 & 0.482759 \end{bmatrix}.$$

The relative error $\|A^{(2)} - X_{23}\|_\infty / \|A^{(2)}\| = \|X_1 - X_{23}\| / \|X_1\|$, where the matrix norm $\| \cdot \|$ represents the maximal singular value of the argument, is equal to $3.40765 \times 10^{-11}$. 

Also, obtained $r_{20} = 2$, which is the exact approximation of $\text{rank}(R) = 2$.

Now, we generate a $[1, 2, 3]$-inverse of $A$. Applying MATHEMATICA function \texttt{QRDecomposition}, we obtain the following full-rank factorization $A = PQ$ for $A$:

$$
P = \begin{bmatrix}
0.122169 & 0.329377 & 0.753171 & -0.229416 \\
0.122169 & 0.7528 & -0.393966 & -0.114708 \\
0.244339 & 0.265331 & 0.382379 & 0.688247 \\
0.366508 & 0.191285 & 0.0115872 & -0.573539 \\
0.488678 & 0.117239 & -0.359205 & 0.344124 \\
0.733017 & -0.444275 & 0.046349 & -0.114708
\end{bmatrix},
$$

$$
Q = \begin{bmatrix}
0 & 2.41883 & 3.64059 & 6.05942 & 0.555344 \\
0 & 0 & 0.440315 & 0.440315 & -0.625711 \\
0 & 0 & 0 & 0 & 0.358831
\end{bmatrix}.
$$

Using $G = P^T$ and the following, randomly chosen $n \times r = 5 \times 4$ matrix

$$
F = \begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 1 \\
-1 & -1 & 2 & 3 \\
0 & 1 & -2 & 3 \\
1 & 2 & 1 & 3
\end{bmatrix},
$$

we obtain

$$
R = FG = \begin{bmatrix}
2.14277 & -0.0129617 & 4.67513 & -1.51032 & 1.02204 & -0.475318 \\
4.04574 & 0.812165 & 3.50243 & 0.779682 & 0.236379 & 0.203896 \\
0.356548 & -2.00703 & 2.31983 & -2.25524 & -0.291955 & -0.540167 \\
-1.85521 & 1.19661 & 1.56531 & -1.55251 & 1.86802 & -0.881097 \\
0.865847 & 0.896979 & 3.22212 & -0.959952 & 1.39632 & -0.453308
\end{bmatrix}.
$$

According to (2.28), the optimal number of iterative steps which ensures the precision $\delta = 10^{-6}$ is equal to $k = 20$. In terms of (2.22) we compute $\beta_{\text{opt}} = 0.000190063$. Corresponding $[1, 2, 3]$-inverse of $A$, obtained after 20 iterations, is

$$
X_{20} = \begin{bmatrix}
0.152778 & -0.541667 & 1.06944 & -0.722222 & 0.194444 & 0.0138889 \\
-0.652778 & 2.29167 & -6.59344 & 4.47222 & -1.44444 & 0.736111 \\
-0.597222 & -0.0416667 & 1.81944 & -1.47222 & 0.944444 & -0.486111 \\
-0.5 & -0.25 & 1.5 & -1.25 & 0.75 & -0.25
\end{bmatrix},
$$

and the approximation of its rank is equal to $r_{20} = 4$. If we apply standard MATHEMATICA function \texttt{MatrixRank[]} to compute the rank of $X$, we obtain $\text{MatrixRank}[X] = 4$, which is approximated by $r_{20}$. Relative error estimate is $\|A^{(1,2,3)} - X_{20}\|/\|A^{(1,2,3)}\| = \|X1 - X_{20}\|/\|X1\| = 1.39537 \times 10^{-11}$.

In the sequel we verify the recommendation that the number of iterations should be as small as possible. After $k = 45 > 20$ iterations we obtain the following $[1, 2, 3]$-inverse of $A$

$$
X_{25} = \begin{bmatrix}
0.151176 & -0.540797 & 1.06935 & -0.722722 & 0.195448 & 0.0135242 \\
-0.651176 & 2.2908 & -6.56935 & 4.47272 & -1.44545 & 0.736476 \\
1.59882 & -1.2092 & 1.18065 & -0.277278 & -0.695448 & 0.236476 \\
-0.598824 & -0.040797 & 1.81935 & -1.47272 & 0.945448 & -0.486476 \\
-0.5 & -0.25 & 1.5 & -1.25 & 0.75 & -0.25
\end{bmatrix}.
$$
Also, the approximation of \( \text{rank}(X_{25}) \) is equal to \( r_{245} = 4.00037 \). \text{MATHEMATICA} function \text{MatrixRank}[] produces incorrect result: \text{MatrixRank}[X_{245}] = 5. Since \text{MatrixRank}[R] = 4, from Corollary 2.3 we conclude that rank of \( X \) is equal to 4, which is approximated by \( r_{245} \). Therefore, we also show that the result proved in Corollary 2.3 is important.

The Moore-Penrose inverse of \( A \) is equal to

\[
Pseudoinverse[A] = A^\dagger = \begin{bmatrix}
4 & -1 & -8 & 7 & -5 & 3 \\
-8 & 15 & -36 & 24 & -5 & 3 \\
10 & -13 & 26 & -15 & 1 & -1 \\
-2 & 3 & -2 & 1 & 1 & -1 \\
-4 & -2 & 12 & -10 & 6 & -2
\end{bmatrix},
\]

where \text{Pseudoinverse}[A] is standard \text{MATHEMATICA} function for computing \( A^\dagger \) (see, for example [20]).

On the other hand, applying iterations (2.14) in the case \( R = FG = A^* \), we obtain an approximation of \( A^\dagger \). In accordance with (2.28), the minimal number of iterative steps, corresponding to \( \delta = 10^{-6} \), is equal to \( k = 36 \), for \( \beta_{\text{opt}} = 4.318278 \times 10^{-5} \). The approximation of the Moore-Penrose inverse of \( A \) is

\[
X_{256} = \begin{bmatrix}
0.500002 & -0.125004 & -0.999991 & 0.874994 & -0.624999 & 0.374999 \\
-1 & 1.575 & -4.50001 & 2.87501 & -0.625001 & 0.375001 \\
1.25 & -1.625 & 3.24999 & -1.87499 & 0.124999 & -0.124999 \\
-0.249997 & 0.374996 & -0.249991 & 0.124994 & 0.125001 & -0.125001 \\
-0.5 & -0.25 & 1.5 & -1.25 & 0.75 & -0.25
\end{bmatrix}.
\]

In this example we again show that information \( r_{2k} \rightarrow \text{rank}(X) \) is important. An application of the standard \text{MATHEMATICA} function \text{MatrixRank}[X_{256}] gives the result 5. However, since \text{MatrixRank}[R] = 4, we conclude that \text{MATHEMATICA} is unable to compute the exact rank of \( X_{256} \). On the other hand, we obtain \( r_{256} = 4 \), which is the desired value in accordance with Corollary 2.3. As a confirmation, we observe that \text{MatrixRank}[\text{Pseudoinverse}[A]] = 4. Relative error estimate is \( \|A^\dagger - X_{256}\|/\|A^\dagger\| = 3.12183 \times 10^{-6} \).

**Example 3.3** Consider the following matrix:

\[
A = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 \\
-1 & -1 & 1 & -1 & 0 & 0 \\
-1 & -1 & -1 & 1 & 0 & 0 \\
-1 & -1 & -1 & -1 & 0 & -1 \\
-1 & -1 & 0 & -1 & -1 & 2
\end{bmatrix}.
\]

Using \( R = A^2 \) and \( \beta = 0.00137174 \), derived from (2.22), SMS method produces well known approximation with the precision \( \delta = 10^{-6} \) of the Drazin inverse from [19] after \( k = 15 \) iterations:

\[
X_{215} = \begin{bmatrix}
0.25 & -0.25 & 0. & 0. & 0. & 0. \\
-0.25 & 0.25 & 0. & 0. & 0. & 0. \\
0. & 0. & 0.25 & -0.25 & 0. & 0. \\
0. & 0. & -0.25 & 0.25 & 0. & 0. \\
0. & 0. & -0.416667 & -0.583333 & 0.666667 & 0.333333 \\
0. & 0. & -0.583333 & 0.416667 & 0.333333 & 0.666667
\end{bmatrix}.
\]
The exact Drazin inverse is equal to

\[
A^D = \begin{bmatrix}
\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 \\
-\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{4} & -\frac{1}{4} & 0 & 0 \\
0 & 0 & -\frac{1}{4} & \frac{1}{4} & 0 & 0 \\
0 & 0 & -\frac{12}{12} & \frac{12}{12} & \frac{1}{3} & \frac{1}{3} \\
0 & 0 & -\frac{12}{12} & \frac{12}{12} & \frac{1}{3} & \frac{1}{3}
\end{bmatrix}.
\]

The relative error \(\|A^D - X_{215}\|/\|A^D\|\), where the matrix norm \(\|\cdot\|\) represents the maximal singular value of the argument, is equal to \(2.67633 \times 10^{-14}\). Let us mention that in [19] Euler-Knopp representation, corresponding to iterations (2.6) in the case \(R = A^l, l \geq \text{ind}(A)\), gives an analogous representation in 29697 iterations.

Rank of the matrix \(R\) can be computed using \(r_{215} = 4\).

**Example 3.4** In this example we verify SMS algorithm (2.14) in the case \(\text{Re} \lambda_i < 0\), for each \(i = 1, \ldots, t\). For this purpose, we accomplished the SMS algorithm for the matrix \(-A\), where \(A\) is defined as in Example 3.1. We used \(\beta_{\text{opt}}^C = -9.208888 \times 10^{-6}\) defined by (2.23) and the number of iterative steps \(k = 23\) conditioned by the second case of (2.28). Corresponding outer inverse, generated by the SMS algorithm is equal to \(-X_{223}\), where \(X_{223}\) is the outer inverse of \(A\) generated in Example 3.1.

### 4 Conclusion

SMS algorithm is investigated in the papers [3], [16], [18]. In these articles SMS algorithm is based on successive squaring of the appropriate composite matrix \(T\) in the form (2.2). In the paper [3] using that \(P = I - \beta A^*A, Q = \beta A^*\) the Moore-Penrose inverse is derived. The weighted Moore-Penrose inverse is derived in [18], using \(P = I - \beta A^TA, Q = \beta A^T\), and finally the Drazin inverse is derived in [16] applying the SMS iterative process with the matrices \(P = I - \beta A^TA, Q = \beta A^T\).

In the present article we introduced and investigated SMS iterative scheme based on the matrices \(P = I - \beta RA\) and \(Q = \beta R\). In this way, we derive an universal SMS algorithm containing all previous results as partial cases. Moreover, we prove that our method can be used to approximate various outer inverses of a given matrix \(A \in \mathbb{C}^{m \times n}\). Properties of the generated outer inverse \(X\) corresponding to chosen matrix \(R\) are examined. Moreover, in the case when a full-rank factorization \(R = FG\) is known, we give an explicit form of an outer generalized inverse of a \(A\) corresponding to the chosen \(R\). An explicit approximation of rank \((R)\) and rank \((X)\) is also given.

We implemented our algorithm in the programming package MATHEMATICA, and many numerical examples are presented in the last section. In these examples we show that sometimes MATHEMATICA is unable to compute the exact value of rank \((X)\), so its iterative approximation is important.
References


