A note on the representation for the Drazin inverse of 2 × 2 block matrices

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Abstract
In this note the representations of the Drazin inverse of a 2 × 2 block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where $A$ and $D$ are square matrices, have been recently developed under some assumptions. We derive formulae for the Drazin inverse of a block matrix $M$ under conditions weaker than those in the papers [9], [11], [12] and [14].

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1 Introduction
Let $A \in \mathbb{C}^{n \times n}$. By $\mathcal{R}(A), \mathcal{N}(A)$ and rank$(A)$ we denote the range, the null space and the rank of matrix $A$, respectively. The smallest nonnegative integer $k$ such that rank$(A^{k+1}) = \text{rank}(A^k)$, denoted by ind$(A)$ or $i_A$, is called the index of $A$. If ind$(A) = k$, there exists a unique matrix $A^d \in \mathbb{C}^{n \times n}$ satisfying the following equations

$$A^{k+1}A^d = A^k, \quad A^dAA^d = A^d, \quad AA^d = A^dA,$$

and $A^d$ is called the Drazin inverse of $A$ (see [1, 4, 10, 16]).

In particular, when ind$(A) \leq 1$, the matrix $A^d$ is called the group inverse of $A$ and denoted by $A^\#$. Clearly, ind$(A) = 0$ if and only if $A$ is nonsingular, and in this case $A^d = A^{-1}$. We denote by $A^\pi = I - AA^d$, the projection on $\mathcal{N}(A^k)$ along $\mathcal{R}(A^k)$, where $k = \text{ind}(A)$.

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The Drazin inverse of a square matrix has various applications in singular differential equations and singular difference equations, Markov chains and iterative methods (see [2, 4, 7, 9, 13, 19]).

Campbell and Meyer [4] posed an open problem to find an explicit representation for the Drazin inverse of a $2 \times 2$ block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in terms of the blocks of the partition, where the blocks $A$ and $D$ are assumed to be square matrices but their sizes need not be the same. Such representations will, in particular, be very useful to find a general expression for the solutions of the second order system of the differential equations (see [2, 3, 12]).

Until now, there has been no explicit formula for the Drazin inverse of $M$ in terms of $A^d$ and $D^d$ with arbitrary $A, B, C$ and $D$. However a general expression for the Drazin inverse of a $2 \times 2$ block triangular matrix (either $B = 0$ or $C = 0$) is presented in the papers of R.E. Hartwig et al.[11] and of C. D. Meyer et al. [14]:

**Theorem 1.1** If $M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$, where $A \in \mathbb{C}^{n \times n}$ and $D \in \mathbb{C}^{m \times m}$, then

$$M^d = \begin{bmatrix} A^d & X \\ 0 & D^d \end{bmatrix},$$

(1)

where

$$X = X(A, B, D) = \sum_{n=0}^{i_D-1} (A^d)^{n+2} BD^n D^\pi + \sum_{n=0}^{i_A-1} A^\pi A^n B (D^d)^{n+2} - A^d BD^d.$$  

(2)

Many other papers have considered this open problem and each of them offered a formula for the Drazin inverse and specific conditions for the $2 \times 2$ block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ to satisfy so that the formula is valid (see [4, 5, 9, 12, 17, 18]).

In the paper of D. Djordjević et al. [9] the representation of $M^d$ is given under the conditions:

$$BC = 0, \ DC = 0 \ and \ BD = 0,$$

while in the paper of R.E. Hartwig et al. [12] the representation of $M^d$ is given under the conditions:

$$BC = 0, \ DC = 0 \ and \ D \ is \ nilpotent.$$
In this paper, using an additive result for the Drazin inverse proved in [6], we derive formulae for the Drazin inverse of a $2 \times 2$ block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, under the conditions weaker than those in the papers [9], [11], [12] and [14].

## 2 Results

First, we present an additive result for the Drazin inverse proved in [6], which we will be useful in proving our main result.

**Theorem 2.1** Let $P, Q \in \mathbb{C}^{n \times n}$ be such that $Q^d = 0$ and the following conditions are satisfied

\[ P^\pi Q = Q, \quad PQP^\pi = 0. \]  

Then,

\[ (P + Q)^d = P^d + \sum_{n=0}^{\infty} (P + Q)^n Q (P^d)^{n+2}. \]

In the following theorem we derive a formula for the Drazin inverse of block-matrix $M$ under some rather cumbersome and complicated conditions but the theorem itself will have a number of useful consequences which will include much simpler conditions.

**Theorem 2.2** Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where $A \in \mathbb{C}^{n \times n}$ and $D \in \mathbb{C}^{m \times m}$. If

\[ D^\pi C = C, \quad BCA^\pi = 0, \quad DCA^\pi = 0 \]  

and

\[ \sum_{n=0}^{iD-1} (A^d)^{n+1} BD^n C = 0, \]
\[ \sum_{n=0}^{iD-1} DC (A^d)^{n+1} BD^n D^\pi = 0, \]
\[ \sum_{n=0}^{iD-1} BC (A^d)^{n+1} BD^n D^\pi = 0. \]
then
\[
M^d = \left[ \begin{array}{cc} A^d & X \\ 0 & D^d \end{array} \right] + \sum_{n=0}^{\infty} \left[ \begin{array}{cc} A & B \\ C & D \end{array} \right]^n \left[ \begin{array}{cc} 0 & 0 \\ C(A^d)^{n+2} & \sum_{i=1}^{n+2} C(A^d)^{i-1}X(D^d)^{n+2-i} \end{array} \right], \quad (6)
\]

where \( X = X(A, B, D) \) is defined by (2).

**Proof.** We rewrite \( M = P + Q \), where \( P = \left[ \begin{array}{cc} A & B \\ 0 & D \end{array} \right] \) and \( Q = \left[ \begin{array}{cc} 0 & 0 \\ C & 0 \end{array} \right] \).

By Theorem 1.1,
\[
P^d = \left[ \begin{array}{cc} A^d & X \\ 0 & D^d \end{array} \right],
\]

where \( X = X(A, B, D) \) is defined by (2). Now, we have that the condition \( P^\pi Q = Q \) is equivalent to
\[
-(AX + BD^d)C = 0,
\]
\[
D^\pi C = C
\]

while the condition \( PQP^\pi = 0 \) is equivalent to
\[
BCA^\pi = 0, DCA^\pi = 0,
\]
\[
-BC(AX + BD^d) = 0,
\]
\[
-DC(AX + BD^d) = 0.
\]

Since,
\[
AX + BD^d = \sum_{n=0}^{i_d-1} (A^d)^{n+1}BD^nD^\pi + \sum_{n=0}^{i_1-1} A^\pi A^nB(D^d)^{n+1},
\]

under the condition (4), we get that conditions (7), (8) and (9) are equivalent to (5). Hence, if (4) and (5) hold, then the conditions from the Theorem 2.1 are satisfied. Now, by Theorem 2.1
\[
M^d = P^d + \sum_{n=0}^{\infty} M^n Q(P^d)^{n+2}
\]
\[
= \left[ \begin{array}{cc} A^d & X \\ 0 & D^d \end{array} \right] + \sum_{n=0}^{\infty} \left[ \begin{array}{cc} A & B \\ C & D \end{array} \right]^n \left[ \begin{array}{cc} 0 & 0 \\ C(A^d)^{n+2} & \sum_{i=1}^{n+2} C(A^d)^{i-1}X(D^d)^{n+2-i} \end{array} \right]. \quad \blacksquare
\]
Note that
\[ S_n = \sum_{i=1}^{n+2} C(A^d)^{i-1} X(D^d)^{n+2-i} = \sum_{j=0}^{i_A-1} C A^j B (D^d)^{n+j+3} \]
\[ + \sum_{j=0}^{i_D-1} C (A^d)^{n+j+3} B D^j D^\pi - \sum_{j=0}^{n+2} C (A^d)^{j} B (D^d)^{n+3-j}. \]

We have that if \( BC = 0 \) and \( DC = 0 \) then the conditions of Theorem 2.2 are satisfied and we get a representation of \( M^d \). So we can see that the condition of Lemma 2.2 of [12] that \( D \) is nilpotent is actually superfluous, as well as the condition \( BD = 0 \) from Theorem 5.3 of [9].

**Corollary 2.1** Let \( M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \), where \( A \in \mathbb{C}^{n \times n} \) and \( D \in \mathbb{C}^{m \times m} \). If \( BC = 0 \) and \( DC = 0 \), then
\[ M^d = \begin{bmatrix} A^d & X \\ C(A^d)^2 & Y + D^d \end{bmatrix}, \]
where \( X = X(A, B, D) \) is defined by (2) and \( Y = CXD^d + CA^dX \).

**Proof.** If \( BC = 0 \) and \( DC = 0 \), it is evident that
\[ \begin{bmatrix} A & B \\ C & D \end{bmatrix}^n \begin{bmatrix} 0 & 0 \\ C(A^d)^{n+2} & \sum_{i=1}^{n+2} C(A^d)^{i-1} X(D^d)^{n+2-i} \end{bmatrix} = 0, \quad n \geq 1, \]
so,
\[ M^d = \begin{bmatrix} A^d & X \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ C(A^d)^2 & CXD^d + CA^dX \end{bmatrix} + 1 \sum_{j=0}^{i_D-1} C(A^d)^{j+1} B (D^d)^{2-j} + \sum_{j=0}^{i_A-1} C A^j B (D^d)^{j+3}. \]

The following theorem presents conditions weaker than those given in Theorem 2.2 under which the representation of \( M^d \) given by (6) is also valid.
Theorem 2.3 Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where $A \in \mathbb{C}^{n \times n}$ and $D \in \mathbb{C}^{m \times m}$. If one of the following conditions is satisfied:

1. $D^\pi C = C$, $BCA^\pi = 0$, $DCA^\pi = 0$, $A^\pi B = B$,
2. $CA^\pi = 0$, $D^\pi C = C$, $A^\pi B = B$,
3. $D^\pi C = C$, $BCA^\pi = 0$, $DCA^\pi = 0$, $A^d BD = 0$, $A^d BC = 0$, $CA^d B = 0$,

then $M^d$ has a representation of the form (6).

Proof. The proof follows directly from the Theorem 2.2.

Using that $M = P + Q$, where $P = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$, we get a similar result as in Theorem 2.2:

Theorem 2.4 Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where $A \in \mathbb{C}^{n \times n}$ and $D \in \mathbb{C}^{m \times m}$. If

$$A^\pi B = B, \quad ABD^\pi = 0, \quad CBD^\pi = 0$$

(10)

and

$$\sum_{n=0}^{i_A-1} AB(D^d)^{n+1} CA^n A^\pi = 0,$$
$$\sum_{n=0}^{i_A-1} CB(D^d)^{n+1} CA^n A^\pi = 0,$$
$$\sum_{n=0}^{i_A-1} (D^d)^{n+1} CA^n B = 0,$$

(11)

then

$$M^d = \begin{bmatrix} A^d & 0 \\ X & D^d \end{bmatrix} + \sum_{n=0}^{\infty} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^n \begin{bmatrix} \sum_{i=0}^{n+1} B(D^d)^i X(A^d)^{n+1-i} & B(D^d)^{n+2} \\ 0 & 0 \end{bmatrix},$$

(12)

where $X = X(D, C, A)$ is defined by (2).
Corollary 2.2 Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where $A \in \mathbb{C}^{n \times n}$ and $D \in \mathbb{C}^{m \times m}$. If $AB = 0$ and $CB = 0$, then

$$M^d = \begin{bmatrix} A^d + Y & B(D^d)^2 \\ X & D^d \end{bmatrix},$$

where $X = X(D, C, A)$ is defined by (2) and $Y = BXA^d + BD^dX$.

Theorem 2.5 Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where $A \in \mathbb{C}^{n \times n}$ and $D \in \mathbb{C}^{m \times m}$. If one of the following conditions is satisfied:

1. $A^\pi B = B, ABD^\pi = 0, CBD^\pi = 0, D^\pi C = C$,
2. $BD^\pi = 0, A^\pi B = B, D^\pi C = C$,
3. $A^\pi B = B, ABD^\pi = 0, CBD^\pi = 0, D^dCA = 0 = 0, D^dCB = 0, BD^dC = 0$,

then $M^d$ can be represented as in (12).

3 Example

The following example describes a $2 \times 2$ matrix $M$ which does not satisfy conditions from Theorems 5.3 and Lemma 2.2, respectively of [9] and [11], whereas the conditions of Theorem 2.2 are met, which allows us to compute $M^D$.

**Example.** Consider a $2 \times 2$ block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = D = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$. Since $BD \neq 0$ and $D$ is not nilpotent, the mentioned results from [9] and [11] fail to apply. It is evident that $BC = 0$ and $DC = 0$, so we can apply Corollary 2.1, thus obtaining

$$M^d = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 \\ -1 & 0 & 2 & 2 \end{bmatrix}. $$
References


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