TOPOLOGICAL EXACT SOLITON SOLUTION OF THE POWER LAW KdV EQUATION

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Abstract

This paper obtains the exact 1-soliton solution of the perturbed Korteweg-de Vries equation with power law nonlinearity. The topological soliton solutions are obtained. The solitary wave ansatz is used to carry out this integration. The domain restrictions are identified in the process and the parameter constraints are also obtained. It has been proved that topological solitons exist only when the KdV equation with power law nonlinearity reduces to simply KdV equation.

AMS Codes: 35Q51, 35Q53, 37K10.
PACS Codes: 02.30.Jr, 02.30.Ik
1 INTRODUCTION

The Korteweg-de Vries (KdV) equation arises in the study of shallow water waves in the context of fluid dynamics. This equation was first derived in 1895 by D. J. Korteweg and G. de Vries to model water waves in a shallow canal. Their goal was to settle a long-standing question: whether a solitary wave could persist under those conditions. Based on his personal observation of such waves since the 1830’s, the naturalist John Scott Russell insisted that such waves do occur, but several prominent mathematicians, including Stokes, were convinced that they were impossible.

Korteweg and de Vries proved that Russell was correct by finding explicit, closed-form, travelling wave solutions to their equation that decay rapidly and thus it represents a highly localized moving hump. Both, the fact that such a solution to a nonlinear equation could exist and the fact that one could write it explicitly were later to be recognized as extremely important, but they went relatively unnoticed at the time.

The KdV equation did not receive significant further attention until 1965, when N. Zabusky and M. Kruskal published the results of their numerical experimentation with the equation. Their computer generated approximate solutions to the KdV equation indicated that any localized initial profile that was allowed to evolve according to KdV equation eventually consisted of a finite set of localized traveling waves of the same shape as the original solitary waves discovered in 1895. Furthermore, when two of the localized disturbances collided, they would emerge from the collision again as another pair of travelling waves with a shift in phase as the only consequence of their interaction. Since the solitary waves made up these solutions seemed to behave like particles, Zabusky and Kruskal coined the term soliton to describe them.

Shortly after, another remarkable discovery was made concerning the KdV equation. It is possible to write many exact solutions to the KdV equation by using ideas from IST. In particular, the exact solution that is discussed in this paper is derivable from IST. In modern terminology, it can be said that this is the discovery of the first integrable nonlinear partial differential equation [1-10].

2 MATHEMATICAL ANALYSIS

The KdV equation with power law nonlinearity is given by

\[ q_t + a q^n q_x + b q_{xxx} = 0 \]  

(1)

Here in (1), \( a \) and \( b \) are constants. They respectively represent the coefficient of nonlinear and dispersion terms. The first term is the evolution term, and thus these equations, mathematically, fall into the category of nonlinear evolution equations. The index \( n \) in the nonlinear term is the index of nonlinearity and is therefore called the power law nonlinearity. The special case where \( n = 1 \), (1) is the KdV equation and when \( n = 2 \), equation (1) is known as the modified KdV (mKdV) equation. It needs to be noted that for the power law KdV equation it is necessary to have \( n \neq 4 \) for soliton solutions to exist. Solitons are the outcome of a delicate balance between dispersion and nonlinearity.

Besides this form of power law KdV equation, there are various other versions of this equation that arises in many areas of Physical Sciences. They are the cylindrical KdV equation [9], Gardner’s equation [2], \( K(m,n) \) equation [8] and many more. These equations have all been studied and they all have been integrated and exact solution has been obtained.

Equation (1) is not integrable by the method of Inverse Scattering Transform that classically integrates the special cases where \( n = 1 \) and \( n = 2 \). In this paper, the focus is going to be on the
perturbed KdV equation that is given by
\[ q_t + a q^n q_x + b q_{xxx} = \gamma q_x q_{xx} + \lambda q_{xxx} + \zeta q_{xxx} q_{xxx} + \theta q_{xx} q_{xxx} + \kappa q_{xxxx} \]  
(2)

These perturbation terms arise in the study of solitary waves in the context of oceanography and shallow water waves [1, 3].

2.1 TOPOLOGICAL SOLITON SOLUTION

In order to look for exact topological 1-soliton solution to the perturbed KdV equation with power law nonlinearity given by (2), the starting hypothesis is given by
\[ q(x, t) = A \tanh^p \tau \]  
(3)
where \( \tau \) is given by (8) and \( v \) is the velocity of the soliton. For the case of topological soliton, \( A \) and \( B \) are known as free parameters. Substituting the ansatz (3) into (1), yields the relation
\[ \begin{align*}
- pvAB \left( \tanh^{p-1} \tau - \tanh^{p+1} \tau \right) \\
+ apA_{p+1}B \left( \tanh^{np+1} \tau - \tanh^{np+1} \tau \right) \\
+ bpAB^3 \left\{ (p-1)(p-2) \tanh^{p-3} \tau - (3p^2 - 3p + 2) \tanh^{p-1} \tau \right. \\
+ \left\{ (3p^2 + 3p + 2) \tanh^{p+3} \tau - (p+1)(p+2) \tanh^{p+3} \tau \right\} \\
= \gamma p^2 A^2 B^5 \left\{ (p-1) \tanh^{2p-3} \tau - (3p-1) \tanh^{2p-1} \tau \right. \\
+ \left( 3p - 1 \right) \tanh^{2p+1} \tau - (p+1) \tanh^{2p+3} \tau \right\} \\
+ \lambda p A^2 B^3 \left\{ (p-1)(p-2) \tanh^{2p-3} \tau - (3p^2 - 3p + 2) \tanh^{2p-1} \tau \right. \\
+ \left( 3p^2 + 3p + 2 \right) \tanh^{2p+1} \tau - (p+1)(p+2) \tanh^{2p+3} \tau \right\} \\
+ \xi p^2 A^2 B^5 \left[ (p-1)(p-2)(p-3) \tanh^{2p-5} \tau \right. \\
- \left( p-1 \right) \left\{ (p+1)(p+2)(p+3) + (3p^2 + 3p + 2) \right\} \tanh^{2p-3} \tau \\
+ \left\{ (p-1)(p-2)(p+1) + 2p - 1 \left( 3p^2 + 3p + 2 \right) + (p+1) \left( 3p^2 + 3p + 2 \right) \right\} \tanh^{2p+1} \tau \\
- \left\{ (p-1)(p+1)(p+2) + 2p - 1 \left( 3p^2 + 3p + 2 \right) + (p+1) \left( 3p^2 + 3p + 2 \right) \right\} \tanh^{2p+3} \tau \\
+ \left( p+1 \right) \left\{ (p+2)(p+3) + (3p^2 + 3p + 2) \right\} \tanh^{2p+5} \tau \\
+ \theta p^2 A^2 B^5 \left[ (p-1)^2(p+2) \tanh^{2p-5} \tau \right. \\
- \left( p-1 \right) \left\{ 2p(p-2) + (3p^2 + 3p + 2) \right\} \tanh^{2p-3} \tau \\
+ \left\{ (p+1)(p-1)(p+2) + (p+1) \left( 3p^2 + 3p + 2 \right) + (p+1) \left( 3p^2 + 3p + 2 \right) \right\} \tanh^{2p+1} \tau \\
- \left\{ (p-1)(p+1)(p+2) + 2p \left( 3p^2 + 3p^2 + 2 \right) + (p+1) \left( 3p^2 + 3p^2 + 2 \right) \right\} \tanh^{2p+3} \tau \\
+ \left( p+1 \right) \left\{ 2p(p+2) + (3p^2 + 3p^2 + 2) \right\} \tanh^{2p+5} \tau \\
+ \kappa p A^2 B^5 \left[ (p-1)(p-2)(p-3)(p-4) \tanh^{2p-5} \tau \right. \\
- \left( p-1 \right)(p-2)(p-3)(p-4) \tanh^{2p-3} \tau \\
- \left( p-1 \right)(p-2) \left\{ (p-2)(p-3) + (3p^2 + 3p^2 + 2) \right\} \tanh^{2p-3} \tau \\
+ \left( p-1 \right)(p-2) \left\{ (p-2)(p-3) + (3p^2 + 3p^2 + 2) \right\} \tanh^{2p-1} \tau \right\}. 
\end{align*} \]
\[
\begin{align*}
&+ p \left\{ (p - 1) \left( 3p^2 - 3p + 2 \right) + (p + 1) \left( 3p^2 + 3p + 2 \right) \right\} \tanh^{2p-1} \tau \\
&- p \left\{ (p - 1) \left( 3p^2 - 3p + 2 \right) + (p + 1) \left( 3p^2 + 3p + 2 \right) \right\} \tanh^{2p+1} \tau \\
&- (p + 1)(p + 2) \left\{ (p + 2)(p + 3) + (3p^2 + 3p + 2) \right\} \tanh^{2p+3} \tau \\
&+ (p + 1)(p + 2)(p + 3)(p + 4) \tanh^{2p+3} \tau \\
&- (p + 1)(p + 2)(p + 3)(p + 4) \tanh^{2p+5} \tau \right] \\
\end{align*}
\] (4)

From (4), equating the exponents \( np + p - 1 \) and \( p + 1 \) gives

\[ np + p - 1 = p + 1 \] (5)

which yields

\[ p = \frac{2}{n} \] (6)

The same value of \( p \) is yielded on equating the exponents \( np + p + 1 \) and \( p + 3 \). In (4), the first set of linearly independent functions are \( \tanh^{p+j} \tau \) for \( j = -1, 1, 3 \). Thus setting their coefficients to zero gives

\[ v = \frac{2bB^2}{n^2} \left( n^2 - 3n + 6 \right) \] (7)

\[ B = \sqrt{-\frac{naA^n}{12b}} \] (8)

and

\[ B = n \sqrt{-\frac{aA^n}{2(n + 1)(n + 2)b}} \] (9)

Equating the two values the free parameter \( B \), from (8) and (9) yields

\[ n = 1 \] (10)

so that from (6)

\[ p = 2 \] (11)

This shows that the topological solitons for the KdV equation with power law nonlinearity will exist only for KdV equation where \( n = 1 \) and not even for the mKdV equation where \( n = 2 \). Thus from (7) and (8), the velocity and the width of the soliton are given by

\[ v = -8bB^2 \] (12)

and

\[ B = \sqrt{-\frac{aA}{12b}} \] (13)

From (12) and (13) it is possible to write

\[ v = \frac{2aA}{3b} \] (14)
Also, from (8), (9) or (13) it can be observed that toplogical solitons will exist for

\[ ab < 0 \]  

(15)

From (4), the other set of linearly independent functions are \( \tanh^{p+l} \tau \) where \( l = -5, -3, -1, 1, 3, 5 \). These respectively yield (11) and

\[ B = \sqrt{-\frac{\gamma}{8(\xi + \theta)}} \]  

(16)

\[ B = \frac{1}{2} \sqrt{\frac{8\lambda + 5\gamma}{34\kappa + 26\theta + 38\xi}} \]  

(17)

\[ B = \frac{1}{2} \sqrt{\frac{10\lambda + 7\gamma}{77\kappa + 29\theta + 47\xi}} \]  

(18)

\[ B = \frac{1}{2} \sqrt{\frac{2\lambda + \gamma}{35\kappa + 11\theta + 15\xi}} \]  

(19)

\[ 15\kappa + 3\theta + 5\xi = 0 \]  

(20)

Now, the relations (16)-(19) impose the restrictions

\[ \gamma(\xi + \theta) < 0 \]  

(21)

\[ (8\lambda + 5\gamma)(34\kappa + 26\theta + 38\xi) > 0 \]  

(22)

\[ (10\lambda + 7\gamma)(77\kappa + 29\theta + 47\xi) > 0 \]  

(23)

\[ (2\lambda + \gamma)(35\kappa + 11\theta + 15\xi) > 0 \]  

(24)

Also, equating the free parameter \( B \) in pairs from (16)-(19) yield the additional constraints between the perturbation coefficients as

\[ (\gamma + 2\lambda)(77\kappa + 29\theta + 47\xi) = (10\lambda + 7\gamma)(35\kappa + 11\theta + 15\xi) \]  

(25)

\[ (\gamma + 2\lambda)(34\kappa + 26\theta + 38\xi) = (8\lambda + 5\gamma)(35\kappa + 11\theta + 15\xi) \]  

(26)

\[ 2(\gamma + 2\lambda)(\xi + \theta) + \gamma(35\kappa + 11\theta + 15\xi) = 0 \]  

(27)

\[ (10\lambda + 7\gamma)(34\kappa + 26\theta + 38\xi) = (8\lambda + 5\gamma)(77\kappa + 29\theta + 47\xi) \]  

(28)

\[ 2(\xi + \theta)(10\lambda + 7\gamma) + \gamma(77\kappa + 29\theta + 47\xi) = 0 \]  

(29)

\[ (\xi + \theta)(8\lambda + 5\gamma) + \gamma(17\kappa + 13\theta + 19\xi) = 0 \]  

(30)
Thus, the velocity of the soliton from (16) and (20)-(23) is given by

\[ v = -\frac{2b(2\lambda + \gamma)}{35\kappa + 11\theta + 15\xi} = -\frac{2b(10\lambda + 7\gamma)}{77\kappa + 29\theta + 47\xi} = -\frac{b(8\lambda + 5\gamma)}{17\kappa + 13\theta + 19\xi} = -\frac{\gamma}{\xi + \theta} \]  

(31)

Hence finally, the equation under consideration, namely (2), reduces to

\[ q_t + aqq_x + bq_{xxx} = \gamma q_xq_{xx} + \lambda qq_{xxx} + \xi q_xq_{xxxx} + \theta q_{xx}q_{xxx} + \kappa q_{xxxx} \]  

(32)

whose topological 1-soliton solution is given by

\[ q(x, t) = A \tanh^2[B(x - vt)] \]  

(33)

where the velocity of the soliton is given by (12) or (31) and the free parameter is given by (13) or (16)-(19). These introduce the restrictions given by (20) and (25)-(29) on the perturbation coefficients and the constraints given by (21)-(24).

3 CONCLUSIONS

This paper integrates the perturbed KdV equation with power law nonlinearity, in presence of perturbation terms. The topological soliton solutions were obtained. An exact 1-soliton solution is obtained together with a number of parameter constraints and domain restrictions on the perturbation coefficients. In addition it was established that topological solitons exist only for KdV equation, which is a very important observation.

ACKNOWLEDGEMENT

The research of the first author (AB) was fully supported by NSF-CREST Grant No: HRD-0630388 and the support is genuinely and sincerely appreciated.
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