Further results on the perfect state transfer in integral circulant graphs

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Abstract

For a given graph G, denote by A its adjacency matrix and $F(t) = \exp(\mathrm{i}At)$. We say that there exists a *perfect state transfer* (PST) in G if $|F(\tau)_{ab}| = 1$, for some vertices a, b and a positive real number τ. Such a property is very important for the modeling of quantum spin networks with nearest-neighbor couplings. We consider the existence of the perfect state transfer in integral circulant graphs (circulant graphs with integer eigenvalues). Some results on this topic have already been obtained by Saxena, Severini, Shparlinski [16], Bašić, Petković & Stevanović [4], and Basić & Petković [3]. In this paper, we show that there exists an integral circulant graph with n vertices having a perfect state transfer if and only if $4 \mid n$. Several classes of integral circulant graphs have been found that have a perfect state transfer for the values of n divisible by 4. Moreover we prove the non-existence of PST for several other classes of integral circulant graphs whose order is divisible by 4. These classes cover the class of graphs where the divisor set contains exactly two elements. The obtained results partially answer the main question of which integral circulant graphs have a perfect state transfer.

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1 Introduction

Quantum spin networks with fixed nearest-neighbor couplings have important applications in quantum information systems. The perfect transfer of quantum states from one qubit to another in such networks was first considered in [6]. The network consists of n qubits where some pairs of qubits are coupled via XY-interaction. There are two special qubits A and B representing the input and output qubit, respectively. The transfer is implemented by setting the qubit A in a prescribed quantum state and by retrieving the state from the output qubit B after some time. The transfer is called *perfect state transfer* (transfer with unit fidelity) if the initial state of qubit A and the final state of qubit B are equal up to a local phase rotation.

Every quantum spin network with fixed nearest-neighbor couplings is uniquely described by an undirected graph G on a vertex set $V(G) = \{1, 2, \ldots, n\}$. The edges of the graph G specify which qubits are coupled. In other words, there is an edge between vertices i and j if i-th and j-th qubit are coupled.

In [6] a simple XY coupling is considered such that the Hamiltonian of the system has the form

$$H_G = \frac{1}{2} \sum_{(i,j)\in E(G)} \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y.$$

and σ_i^x, σ_i^y and σ_i^z are Pauli matrices acting on i-th qubit. The standard basis chosen for an individual qubit is $\{|0\rangle, |1\rangle\}$ and it is assumed that all spins initially point down along the prescribed z axis. In other words, the initial state of the network is $|\underline{0}\rangle = |0_A 0 \ldots 0 0_B\rangle$. This is an eigenstate of Hamiltonian H_G corresponding

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to zero energy. The Hilbert space $\mathcal{H}_G$ associated with a network is spanned by vectors $|e_1e_2\ldots e_n\rangle$ where $e_i \in \{0,1\}$ and, therefore, its dimension is 2^n.

The process of transmitting a quantum state from A to B begins with the creation of the initial state $\alpha|0_A0\ldots00_B\rangle+\beta|1_A0\ldots00_B\rangle$ of the network. Since $|\underline{0}\rangle$ is zero-energy eigenstate of H_G, the coefficient α will not change in time. Since the operator of total z component of the spin $\sigma^z_{tot} = \sum_{i=1}^n \sigma^z_i$ commutes with H_G, state $|1_A0\ldots00_B\rangle$ must evolve into the superposition of the states $|i\rangle = |0\ldots01_i0,\ldots,0\rangle$ for $i = 1,\ldots,n$. Denote by $\mathcal{S}_G$ the subspace of $\mathcal{H}_G$ spanned by vectors $|i\rangle$, $i = 1,\ldots,n$. Hence, the initial state of network evolves in time t into the state

$$\alpha|\underline{0}\rangle + \sum_{i=1}^{n} \beta_i(t)|i\rangle \in \mathcal{S}_G.$$

Previous equation shows that system dynamics is completely determined by the evolution in n-dimensional space $\mathcal{S}_G$. Restriction of the Hamiltonian H_G to subspace $\mathcal{S}_G$ is $n\times n$ matrix identical to adjacency matrix A_G of graph G.

Thus, the time evolution operator can be written in the form $F(t) = \exp(\mathrm{i}A_Gt)$. The matrix exponential $\exp(M)$ is defined as usual

$$\exp(M) = \sum_{n=0}^{+\infty} \frac{1}{n!}M^n.$$

A *perfect state transfer* (PST) between different vertices (qubits) a and b ($1 \le a,b \le n$) is obtained in time τ, if $\langle a|F(t)|b\rangle = |F(\tau)_{ab}| = 1$. The graph (network) is *periodic* at a if $|F(\tau)_{aa}| = 1$ for some τ. A graph is *periodic* if it is periodic at each vertex a.

The existence of PST for some network topologies is already considered in the literature. For example, Christandl, Datta, Dorlas, Ekert, Kay and Landahl [7] prove that PST occurs in the paths of length one and two between its end-vertices and also in Cartesian powers of these graphs between vertices at maximal distance. In the recent paper [10], Godsil constructed a class of distance-regular graphs of diameter three, with PST. Some properties of quantum dynamics on circulant graphs were studied in [1]. Saxena, Severini and Shparlinski [16] considered circulant graphs as the potential candidates for modeling quantum spin networks having PST. They show that a circulant graph is periodic if and only if all eigenvalues of the graph are integers (i.e. graph is integral). Since the periodicity is a necessary condition for PST existence [16], circulant graphs having PST must be *integral circulant graphs*. A simple and general characterization of the existence of PST in an integral circulant graph, in terms of its eigenvalues, is given by Bašić, Petković and Stevanović in [4]. Furthermore, it is shown that for odd number of vertices, there is no PST, and that among the class of unitary Cayley graphs (subclass of integral circulant graphs), only K_2 (complete graph with two nodes) and C_4 (cycle of length four) have PST. In the recent paper [3], the present authors proved that an integral circulant graph with a square-free number of vertices does not have PST. Two classes of integral circulant graphs having PST were also found.

The term 'integral circulant graph' first appears in the work of So [17], where a nice characterization of these graphs in terms of their symbol set is given. The upper bounds on the number of vertices and the diameter of integral circulant graphs are given in [16]. Various other properties of unitary Cayley graphs were recently investigated. For example, Berrizbeitia and Giudici [5] and Fuchs [8] established the lower and upper bound on the size of the longest induced cycles. Klotz and Sander [12] determined the diameter, clique number, chromatic number and eigenvalues of unitary Cayley graphs. Bašić and Ilić [2] calculated the clique number of integral circulant graphs with exactly one and two divisors and also provided the inequality for the general case.

This paper extends the results from papers [3, 4, 16]. First it is proven that there exists an integral circulant graph with n vertices having PST if and only if $4 \mid n$. Several classes of integral circulant graphs having PST for different values of n are also found. It is proved that an integral circulant graph, where the divisor set consists of only odd divisors, has no PST. Moreover, we prove the non-existence of PST for several other classes of integral circulant graphs whose order is divisible by 4. These classes cover the class of integral circulant graphs where the divisor set contains exactly two elements. These results partially answer the main question of which integral circulant graphs have PST?

2 Integral circulant graphs

A *circulant graph* $G(n; S)$ is a graph on vertices $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ such that each vertex i is adjacent to vertices $i +_n s$ for all $s \in S$. A set $S \subseteq \mathbb{Z}_n$ is called the *symbol* of graph $G(n; S)$ and $+_n$ denotes addition over modulo n. As we will consider undirected graphs without loops, we assume that $S = n - S = \{n - s \mid s \in S\}$ and $0 \notin S$. Note that the degree of graph $G(n; S)$ is $\#S$. A graph is *integral* if all its eigenvalues are integers. Wasin So has characterized integral circulant graphs [17] by the following theorem:

Theorem 1 **[17]** *A circulant graph $G(n; S)$ is integral if and only if*

$$S = \bigcup_{d \in D} G_n(d),$$

for some set of divisors $D \subseteq D_n$. Here $G_n(d) = \{k \ : \ \gcd(k, n) = d, \ 1 \le k \le n-1\}$, and D_n is the set of all divisors of n, different from n.

Therefore an *integral circulant graph* (ICG) $G(n; S)$ is defined by its order n and the set of divisors D. Such graphs are also known as *gcd-graphs* (see for example [12]). An integral circulant graph with n vertices, defined by the set of divisors $D \subseteq D_n$ will be denoted by $\mathrm{ICG}_n(D)$. From Theorem 1 we have that the degree of an integral circulant graph is $\deg \mathrm{ICG}_n(D) = \sum_{d \in D} \varphi(n/d)$. Here $\varphi(n)$ denotes the Euler-phi function [11].

The eigenvalues and eigenvectors of $\mathrm{ICG}_n(D)$ are given in [16] as

$$\lambda_j = \sum_{s \in S} \omega_n^{js}, \quad v_j = [1 \ \omega_n^s \ \omega_n^{2s} \cdots \omega_n^{(n-1)s}], \tag{1}$$

where $\omega_n = \exp(\mathrm{i}2\pi/n)$ is the n-th root of unity. Denote by $c(n, j)$ the following expression

$$c(j, n) = \mu(t_{n,j}) \frac{\varphi(n)}{\varphi(t_{n,j})}, \quad t_{n,j} = \frac{n}{\gcd(n, j)}, \tag{2}$$

where μ is the Möbius function. The expression $c(j, n)$ is known as the *Ramanujan function* [11]. Eigenvalues λ_j can be expressed in terms of the Ramanujan function as follows ([12], Theorem 16)

$$\lambda_j = \sum_{d \in D} c(j, n/d). \tag{3}$$

Let us observe the following properties of the Ramanujan function. These basic properties will be used in the rest of the paper.

Proposition 2 *For any positive integers n, j and d such that $d \mid n$, holds*

$$c(0, n) = \varphi(n), \tag{4}$$

$$c(1, n) = \mu(n), \tag{5}$$

$$c(2, n) = \begin{cases} \mu(n), & n \in 2\mathbb{N} + 1 \\ \mu(n/2), & n \in 4\mathbb{N} + 2 \\ 2\mu(n/2), & n \in 4\mathbb{N} \end{cases} \tag{6}$$

$$c(n/2, n/d) = \begin{cases} \varphi(n/d), & d \in 2\mathbb{N} \\ -\varphi(n/d), & d \in 2\mathbb{N} + 1 \end{cases} \tag{7}$$

Proof. Directly using relation (2). $\square$

The integral circulant graph $\mathrm{ICG}_n(D)$ is connected if and only if $\gcd(n, d_1, \ldots, d_k) = 1$ where $D = \{d_1, \ldots, d_k\}$. In the rest of the paper we will only consider connected integral circulant graphs.

3 Perfect state transfer

Let G be an undirected graph and denote by A_G its adjacency matrix. Let $F(t) = \exp(iA_G t)$. There is a *perfect state transfer* (PST) in graph G [6, 10, 16] if there are distinct vertices a and b and a positive real number t such that $|F(t)_{ab}| = 1$.

Let $\lambda_0, \lambda_2, \ldots, \lambda_{n-1}$ be eigenvalues (not necessary distinct) of matrix A_G and $u_0, u_1, \ldots, u_{n-1}$ be corresponding normalized eigenvectors. We use spectral decomposition of the real symmetric matrix A_G (see for example [9] (Theorem 5.5.1) for more details). The matrix function $F(t)$ can be represented as

$$F(t) = \sum_{k=0}^{n-1} \exp(\mathrm{i}\lambda_k t) u_k u_k^*. \tag{8}$$

Now let $G = \mathrm{ICG}_n(D)$ be an integral circulant graph. By simple calculation and using (1), it holds that $\|v_k\| = \sqrt{n}$ and hence $u_k = v_k/\sqrt{n}$. Expression (8) now becomes

$$F(t) = \frac{1}{n} \sum_{k=0}^{n-1} \exp(\mathrm{i}\lambda_k t) v_k v_k^*.$$

Specially, from the last expression and (1) it directly follows

$$F(t)_{ab} = \frac{1}{n} \sum_{k=0}^{n-1} \exp(\mathrm{i}\lambda_k t) \omega_n^{k(a-b)}.$$

This expression is given in [16] (Proposition 1). Finally, our goal is to check whether there exist distinct integers $a, b \in \mathbb{Z}_n$ and a positive real number t such that $|F(t)_{ab}| = 1$, i.e.

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} \exp(\mathrm{i}\lambda_k t) \omega_n^{k(a-b)} \right| = 1. \tag{9}$$

Since the left-hand side of (9) depends on a and b only as the function of $a - b$ we can, without loss of generality, assume that $b = 0$. Therefore, in the rest of the paper we consider the existence of PST only between vertices a and 0.

We restate some results proved in [4]. These results establish necessary and sufficient conditions for (9).

Theorem 3 [4] *There exists a PST in graph* $\mathrm{ICG}_n(D)$ *between vertices* a *and* 0 *if and only if there are integers* p *and* q *such that* $\gcd(p, q) = 1$ *and*

$$\frac{p}{q}(\lambda_{j+1} - \lambda_j) + \frac{a}{n} \in \mathbb{Z}, \tag{10}$$

for all $j = 0, \ldots, n-2$.

Theorem 4 [4] *There is no PST in* $\mathrm{ICG}_n(D)$ *if* n/d *is odd for every* $d \in D$*. For* n *being even, if there exists a PST in* $\mathrm{ICG}_n(D)$ *between vertices* a *and* 0*, then* $a = n/2$.

According to Theorem 4, PST may exists in $\mathrm{ICG}_n(D)$ only between vertices $n/2$ and 0 (i.e., between b and $n/2 + b$ as mentioned in [16]). Hence we will avoid referring to the input and output vertex and will just say that there exists a PST in $\mathrm{ICG}_n(D)$.

The next corollary is derived from Theorem 3 and is further used as the criterion for the nonexistence of PST.

Corollary 5 [4] *If* $\lambda_j = \lambda_{j+1}$ *for some* $j = 0, \ldots, n-2$ *then there is no PST in* $\mathrm{ICG}_n(D)$.

For a given prime number p and integer $n \in \mathbb{N}_0$, denote by $S_p(n)$ the maximal number α such that $p^\alpha \mid n$ if $n \in \mathbb{N}$, and $S_p(0) = +\infty$ for an arbitrary prime number p. The following result is proven in [4] and is further used as the criterion for the existence of PST.

Lemma 6 [4] *There exists PST in* $\mathrm{ICG}_n(D)$, *if and only if there exists a number* $m \in \mathbb{N}_0$ *such that the following holds for all* $j = 0, 1, \ldots, n-2$

$$S_2(\lambda_{j+1} - \lambda_j) = m. \tag{11}$$

The following corollary follows directly from Lemma 6.

Corollary 7 *Let* $\mathrm{ICG}_n(D)$ *have PST. One of the following two statements must hold*

1. $\lambda_j \equiv \lambda_{j+1} \pmod 2$ *for every* $0 \leq j \leq n-1$ *(i.e., all eigenvalues* λ_j *have the same parity).*

2. $\lambda_j \equiv \lambda_{j+1} + 1 \pmod 2$ *for every* $0 \leq j \leq n-1$ *(i.e.,* λ_j *are alternatively odd and even).*

We end this section with the following result concerning unitary Cayley graphs.

Theorem 8 [4] *The only unitary Cayley graphs that have PST are* K_2 *and* C_4.

Therefore, in the rest of the paper we will suppose that a set D contains at least two divisors, i.e. $|D| \geq 2$.

4 Non-existence of PST in $\mathrm{ICG}_n(D)$ when $n \in 4\mathbb{N} + 2$

In this section we prove that there are no integral circulant graphs $\mathrm{ICG}_n(D)$ having PST whose order n satisfies $n \in 4\mathbb{N} + 2$ [1]. This is done in Theorem 13.

In the proof of Theorem 13 we distinguish two cases depending on whether $n/2 \in D$ or not. If $n/2 \notin D$ we show that PST existence implies that all eigenvalues λ_j must be even. According to Lemma 12, the set of divisors D must be equal to $D = D' \cup 2D'$ where $D' \subset D$ is set of odd divisors from D. Using Lemma 11 we connect eigenvalues of graphs $\mathrm{ICG}_n(D)$ and $\mathrm{ICG}_{n/2}(D')$ and provide the contradiction with Lemma 6.

The case $n/2 \in D$ is reduced to the previous case, by establishing connection between eigenvalues of graphs $\mathrm{ICG}_n(D)$ and $\mathrm{ICG}_n(D \setminus \{n/2\})$. Lemma 10 is auxiliary for Lemma 12, which is essential in the proof of the Theorem 13. Lemma 9 is also used in the following sections.

Lemma 9 *Let* $n \geq 2$ *and* $0 \leq j \leq n-1$ *be arbitrary integers such that* $4 \nmid n$ *and let* $n = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ *be the prime factorization of* n *(*$0 \leq \alpha_0 \leq 1$*). It holds that* $c(j, n) \in 2\mathbb{N} + 1$ *if and only if* $j = p_1^{\alpha_1 - 1} \cdots p_k^{\alpha_k - 1} J$ *for some integer* J *such that* $\gcd(J, n) \in \{1, 2\}$.

Proof.

($\Rightarrow$:) Suppose that $c(j, n)$ is an odd integer. Since $c(j, n) = \mu(t_{n,j})\varphi(n)/\varphi(t_{n,j})$, it holds that $\mu(t_{n,j}) = \pm 1$, i.e. $t_{n,j}$ is square-free and $\varphi(n)/\varphi(t_{n,j})$ is an odd integer.

Suppose that for some p_i it holds that $p_i \nmid t_{n,j}$. Let $n' = n/p_i^{\alpha_i}$. Since $t_{n,j} \mid n'$ and $\varphi(t_{n,j}) \mid \varphi(n')$ we obtain that

$$c(j, n) = \frac{\varphi(n)}{\varphi(t_{n,j})} = \frac{\varphi(p_i^{\alpha_i})\varphi(n')}{\varphi(t_{n,j})} = p_i^{\alpha_i - 1}(p_i - 1)\frac{\varphi(n')}{\varphi(t_{n,j})}.$$

The last equation implies that $c(j, n)$ is even since $p_i - 1$ is even. This is a contradiction and we can conclude that $p_i \mid t_{n,j}$ for every $i = 1, 2, \ldots, k$.

If $n \in 2\mathbb{N} + 1$ it must hold that $t_{n,j} = p_1 \cdots p_k$ since $t_{n,j}$ is square-free. If $n \in 4\mathbb{N} + 2$, we have two possibilities for $t_{n,j}$, $t_{n,j} = p_1 \cdots p_k$ or $t_{n,j} = 2p_1 \cdots p_k$ depending on the parity of j.

Furthermore, using $n = \gcd(n, j)t_{n,j}$ we obtain that $\gcd(n, j) = p_1^{\alpha_1 - 1} \cdots p_k^{\alpha_k - 1}$ ($t_{n,j}$ and n have the same parity) or $\gcd(n, j) = 2p_1^{\alpha_1 - 1} \cdots p_k^{\alpha_k - 1}$ (otherwise). Now the condition of the lemma follows straightforward.

($\Leftarrow$:) Since $\gcd(n, j) = p_1^{\alpha_1 - 1} \cdots p_k^{\alpha_k - 1} \gcd(J, n)$, it holds that $t_{n,j} = p_1 \cdots p_k$ or $t_{n,j} = 2p_1 \cdots p_k$. In both cases it holds that $\varphi(t_{n,j}) = (p_1 - 1) \cdots (p_k - 1)$. Now since

$$c(j, n) = \mu(t_{n,j})\frac{\varphi(n)}{\varphi(t_{n,j})} = \mu(t_{n,j})p_1^{\alpha_1 - 1} \cdots p_k^{\alpha_k - 1},$$

we conclude that $c(j, n) \in 2\mathbb{N} + 1$. $\square$

Denote by d_{max} the maximal divisor in the divisor set D.

[1] By $a\mathbb{N} + b$ we denote a set of all numbers $n \in \mathbb{N}$ such that $n \equiv b \pmod a$.

Lemma 10 *Let n be an arbitrary integer such that $4 \nmid n$. Suppose that one of the following conditions is satisfied*

(1) *d_{max} is odd,*

(2) *d_{max} is even and $d_{max}/2 \notin D$.*

Then there exists an odd number $0 \leq j \leq n-1$ such that $c(n/d_{max}, j)$ is odd and $c(n/d, j)$ is even for any $d \in D \setminus \{d_{max}\}$. Furthermore, the eigenvalue λ_j of $\mathrm{ICG}_n(D)$ is odd.

Proof. Consider the prime factorization of n in the form $n = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where $0 \leq \alpha_0 \leq 1$. Since d_{max} is the divisor of n, it can be represented in the form $d_{max} = 2^{\beta_0} p_1^{\beta_1} \cdots p_k^{\beta_k}$ where $0 \leq \beta_i \leq \alpha_i$ for any $i = 1, \ldots, k$. Without loss of generality, we can suppose that there exists $1 \leq s \leq k$ such that $\beta_i < \alpha_i$ for $i = 1, \ldots, s$ and $\beta_i = \alpha_i$ for $i = s+1, \ldots, k$. Then we can write $n/d_{max} = 2^{\alpha_0 - \beta_0} p_1^{\alpha_1 - \beta_1} \cdots p_s^{\alpha_s - \beta_s}$.

Denote by
$$j_0 = p_1^{\alpha_1 - \beta_1 - 1} \cdots p_s^{\alpha_s - \beta_s - 1} p_{s+1}^{\alpha_{s+1}} \cdots p_k^{\alpha_k}.$$
It trivially holds that $0 \leq j_0 \leq n-1$. Lemma 9 directly yields that $c(j_0, n/d_{max})$ is odd since $n/d_{max} \geq 2$.

Let $d \in D \setminus \{d_{max}\}$ be an arbitrary divisor with its prime factorization $d = 2^{\gamma_0} p_1^{\gamma_1} \cdots p_k^{\gamma_k}$, where $0 \leq \gamma_0 \leq 1$

We will show that there exists $1 \leq i \leq k$ such that $0 \leq \gamma_i < \beta_i \leq \alpha_i$. Suppose that this is not valid. If the condition **(1)** is satisfied then $d_{max} \mid d$, which is a contradiction. Similarly, if the condition **(2)** is satisfied, then it holds that $d_{max} \mid 2d$, which implies $d = d_{max}/2$, providing a contradiction with the second part of the condition **(2)**.

Suppose that $\alpha_i - \beta_i \geq 1$. Then $i \leq s$ and $S_{p_i}(n/d) = \alpha_i - \gamma_i \geq 2$. Since $S_{p_i}(j_0) = \alpha_i - \beta_i - 1 > \alpha_i - \gamma_i - 1$, from Lemma 9 we can conclude that $c(j_0, n/d)$ is even.

Now suppose that $\alpha_i = \beta_i$. Then $i > s$ and $S_{p_i}(n/d) = \alpha_i - \gamma_i \geq 1$. Again since $S_{p_i}(j_0) = \alpha_i > \alpha_i - \gamma_i - 1$, Lemma 9 yields that $c(j_0, n/d)$ is even.

The last statement of the lemma now holds trivially since
$$\lambda_{j_0} = c(j_0, n/d_{max}) + \sum_{d \in D \setminus \{d_{max}\}} c(j_0, n/d).$$
$\square$

In the rest of the section, if not stated otherwise, we consider an integral circulant graph $\mathrm{ICG}_n(D)$ with order $n \in 4\mathbb{N} + 2$.

Lemma 11 *If $d \in D$ is an even divisor such that $d/2 \in D$ and $0 \leq j \leq n-1$ is an arbitrary integer, then $c(j, n/d) = -c(j, 2n/d)$ for $j \in 2\mathbb{N}+1$ and $c(j, n/d) = c(j, 2n/d)$ for $j \in 2\mathbb{N}$.*

Proof. Suppose that $j \in 2\mathbb{N}+1$. Then $\gcd(2n/d, j) = \gcd(n/d, j)$ holds and
$$t_{2n/d,j} = \frac{2n}{d \gcd(2n/d, j)} = 2\frac{n}{d \gcd(n/d, j)} = 2t_{n/d,j}.$$
Furthermore, it holds that $\varphi(t_{2n/d,j}) = \varphi(t_{n/d,j})$ and $\varphi(2n/d) = \varphi(n/d)$, since $t_{n/d,j}$ is odd and d is even. Also we have that $t_{2n/d,j}$ is square-free if and only if $t_{n/d,j}$ is square-free, and $\mu(t_{2n/d,j}) = -\mu(t_{n/d,j})$. Now we can directly conclude that
$$c(j, 2n/d) = \mu(t_{2n/d,j}) \frac{\varphi(2n/d)}{\varphi(t_{2n/d,j})} = -\mu(t_{n/d,j}) \frac{\varphi(n/d)}{\varphi(t_{n/d,j})} = -c(j, n/d).$$

Now suppose that $j \in 2\mathbb{N}$. We have $\gcd(2n/d, j) = 2\gcd(n/d, j)$ and also $t_{2n/d,j} = t_{n/d,j}$. This directly yields that $c(j, 2n/d) = c(j, n/d)$. $\square$

Lemma 12 *All eigenvalues λ_j are even if and only if for every odd $d \in D$ it holds that $2d \in D$ and for every even $d \in D$ it holds that $d/2 \in D$.*

Proof.

($\Leftarrow$:) Let $j \in 2\mathbb{N}+1$. According to Lemma 11 we have $c(j, n/d) = -c(j, 2n/d)$ for any even $d \in D$ and therefore $\lambda_j = 0$. For $j \in 2\mathbb{N}$, since $c(j, n/d) = c(j, 2n/d)$ (Lemma 11) for every even $d \in D$, it follows that $\lambda_j = 2\sum_{d\in D\cap 2\mathbb{N}} c(j, n/d) \in 2\mathbb{N}$.

($\Rightarrow$:) Suppose that the statement of the lemma is not valid. Denote by D'' the set of all divisors $d \in D$ such that $2d \in D$ for d odd and $d/2 \in D$ for d even. Let $D' = D \setminus D''$. By the assumption, D' is a non-empty set. According to Lemma 11 it holds that $c(j, n/d) + c(j, 2n/d) \in 2\mathbb{N}$ for every $d \in D''$. Consider now the integral circulant graph $\mathrm{ICG}_n(D')$ and denote $d'_{max} = \max D'$. Set D' satisfies the conditions of Lemma 10. This implies that there is an odd index j_0 such that $c(j_0, n/d'_{max})$ is odd and $c(j_0, n/d)$ is even for every $d \in D' \setminus \{d'_{max}\}$. Now we have

$$\lambda_{j_0} = c(j_0, n/d'_{max}) + \sum_{d\in D'\setminus\{d'_{max}\}} c(j_0, n/d) + \sum_{d\in D''} c(j_0, n/d) \in 2\mathbb{N}+1,$$

since both sums in the last expression are even. This is a contradiction. $\square$

Note that the second condition in Lemma 12 is equivalent to $D = D' \cup 2D'$, where $D' = D \cap 2\mathbb{N}+1$. Now we are ready to prove the following main result of this section.

Theorem 13 *There is no PST in* $\mathrm{ICG}_n(D)$ *for an arbitrary set of divisors* D*, if* $n \in 4\mathbb{N}+2$.

Proof. Denote by D_1 the set of all divisors $d \in D$ such that n/d is a square-free integer. Also denote by $D_e = D_1 \cap 2\mathbb{N}$ and $D_o = D_1 \cap 2\mathbb{N}+1$. It holds that $\lambda_1 = \sum_{d\in D_1} \mu(n/d)$. Since $\lambda_2 = \sum_{d\in D_e} \mu(n/d) + \sum_{d\in D_o} \mu(n/(2d))$, we have that $\lambda_2 - \lambda_1 = \sum_{d\in D_o}(\mu(n/(2d)) - \mu(n/d))$. Since both n/d and $n/(2d)$ are square-free, we have that $2 \mid \lambda_2 - \lambda_1$ or, equivalently, $S_2(\lambda_2 - \lambda_1) \geq 1$. We will distinguish the following two cases.

Case 1. $n/2 \notin D$. Then $\lambda_0 \in 2\mathbb{N}$, according to the Proposition 2.

Suppose that $|D_1|$ is odd. Then λ_1 is also odd and $S_2(\lambda_1 - \lambda_0) = 0$. According to Lemma 6 there is no PST in $\mathrm{ICG}_n(D)$.

Now suppose that $|D_1|$ is even, which implies that λ_1 is even. Suppose that $\mathrm{ICG}_n(D)$ has a PST. According to Corollary 7, all eigenvalues λ_j, $0 \leq j \leq n-1$ are even. But now Lemma 12 implies that $D = D' \cup 2D'$, where $D' = D \cap 2\mathbb{N}+1$. Consider now a new integral circulant graph $\mathrm{ICG}_{n_1}(D')$ where $n_1 = n/2$. Denote its eigenvalues by λ'_j for $j = 0, \ldots, n_1 - 1$. Since n_1/d is odd for every $d \in D'$, we can conclude that $c(2j, n_1/d) = c(j, n_1/d)$ for every $0 \leq j \leq n_1 - 1$. It also holds from Lemma 11 that $c(2j, n/d) = c(2j, n/(2d))$ for any $d \in D'$. All these considerations yield to

$$\begin{aligned}\lambda_{2j} &= \sum_{d\in D} c(2j, n/d) = 2\sum_{d\in D'} c(2j, n/(2d)) \\ &= 2\sum_{d\in D'} c(2j, n_1/d) = 2\sum_{d\in D'} c(j, n_1/d) = 2\lambda'_j.\end{aligned}$$

Since $n_1/2 \notin D'$ it holds that $\lambda'_0 \in 2\mathbb{N}$ which implies $S_2(\lambda_0) \geq 2$. On the other side, according to Lemma 10 there exists j_0 such that λ'_{j_0} is odd. Then $S_2(\lambda_{2j_0}) = 1$.

If $0 \leq j \leq n-1$ is odd, according to Lemma 11 we have $c(j, n/(2d)) = -c(j, n/d)$ for any odd $d \in D$, which implies $\lambda_j = 0$.

Now it holds that $S_2(\lambda_1 - \lambda_0) \geq 2$ and $S_2(\lambda_{2j_0} - \lambda_{2j_0-1}) = 1$, which is in contradiction with Lemma 6.

Case 2. $n/2 \in D$. Now $\lambda_0 \in 2\mathbb{N}+1$.

Suppose that $|D_1|$ is even. Then λ_1 is also even and $S_2(\lambda_1 - \lambda_0) = 0$. Since $S_2(\lambda_2 - \lambda_1) \geq 1$, according to Lemma 6 there is no PST in $\mathrm{ICG}_n(D)$.

Suppose that $|D_1|$ is odd, which implies that λ_1 is also odd. Suppose that there is a PST in $\mathrm{ICG}_n(D)$. According to Corollary 7, all eigenvalues λ_j, for $0 \leq j \leq n-1$ are odd. Let $D' = D \setminus \{n/2\}$. Denote by λ'_j eigenvalues of integral circulant graph $\mathrm{ICG}_n(D')$. Since

$$t_{2,j} = \frac{2}{\gcd(2,j)} = \begin{cases} 2, & 2 \nmid j \\ 1, & 2 \mid j \end{cases}, \quad c(j,2) = \begin{cases} -1, & 2 \nmid j \\ 1, & 2 \mid j \end{cases}. \tag{12}$$

it holds that

$$\lambda_j = \begin{cases} \lambda'_j + 1, & 2 \mid j \\ \lambda'_j - 1, & 2 \nmid j \end{cases}.$$

The graph $\mathrm{ICG}_n(D')$ has all even eigenvalues and $n/2 \notin D'$. Lemma 12 yields that $D' = D'_1 \cap 2D'_1$ where $D'_1 = D' \cap 2\mathbb{N}+1$. Analogously to **Case 1**, we can prove that there exists an odd j_0 such that $S_2(\lambda'_1 - \lambda'_0) \geq 2$ and $S_2(\lambda'_{2j_0} - \lambda'_{2j_0-1}) = 1$. This implies $S_2(\lambda_1 - \lambda_0) = S_2(\lambda'_1 - \lambda'_0 + 2) = 1$ and $S_2(\lambda_{2j_0} - \lambda_{2j_0-1}) = S_2(\lambda'_{2j_0} - \lambda'_{2j_0-1} + 2) \geq 2$, which is in contradiction with Lemma 6. □

5 Classes of ICG graphs having PST for $4 \mid n$

In this section we introduce several classes of integral circulant graphs having a PST. Moreover, we prove that for every n divisible by 4, there exists at least one graph $\mathrm{ICG}_n(D)$ which has a PST.

By direct computation we show that

$$t_{4,j} = \frac{4}{\gcd(4,j)} = \begin{cases} 4, & 2 \nmid j \\ 2, & S_2(j) = 1 \\ 1, & 4 \mid j \end{cases}, \quad c(j,4) = \begin{cases} 0, & 2 \nmid j \\ -2, & S_2(j) = 1 \\ 2, & 4 \mid j \end{cases}. \tag{13}$$

The last equation will be referred to several times in this section.

5.1 Case $8 \mid n$

First we deal with the case $S_2(n) \geq 3$ and find three classes of integral circulant graphs having PST. The following lemma will be referred to several times in this subsection.

Lemma 14 *Let $m \in 8\mathbb{N}+4$ and $m > 4$. There exist two integers $j_1, j_2 \in 4\mathbb{N}+2$ such that $0 \leq j_1, j_2 \leq m-1$ and it holds $c(j_1, m) \in 4\mathbb{N}$ and $c(j_2, m) \in 4\mathbb{N}+2$.*

Proof. Denote $m = 4m_1$ where m_1 is an odd integer. Let $j \in 4\mathbb{N}+2$. Then

$$t_{n,j} = \frac{4m_1}{\gcd(j, 4m_1)} = \frac{2m_1}{\gcd(j/2, 2m_1)} = t_{2m_1, j/2}.$$

Since $\varphi(m) = 2\varphi(2m_1)$, using the previous equation we find that

$$c(j,m) = \mu(t_{m,j}) \frac{\varphi(m)}{\varphi(t_{m,j})} = \mu(t_{2m_1,j/2}) \frac{2\varphi(2m_1)}{\varphi(t_{2m_1,j/2})} = 2c(j/2, 2m_1). \tag{14}$$

Denote by p an arbitrary odd prime divisor of m_1 (since $m_1 > 1$ there exists at least one) and $\alpha = S_p(m_1)$. Let $j_1 = 2p^\alpha$. Now $c(j_1/2, 2m_1)$ is even according to Lemma 9 and using (14) we obtain $c(j_1, m) \in 4\mathbb{N}$.

Let j_2 be an arbitrary integer such that $0 \leq j_2 \leq m-1$, $j_2 \in 4\mathbb{N}+2$ and $\gcd(j_2, m_1) = 1$. Then $c(j_2/2, 2m_1)$ is odd according to Lemma 9. Furthermore using (14) we obtain $c(j_2, m) \in 4\mathbb{N}+2$. □

In the following two theorems we prove that graphs $\mathrm{ICG}_n(\{1, n/4\})$ and $\mathrm{ICG}_n(\{1, n/2\})$ have PST for every n divisible by 8. If n is divisible by 16, we prove that $\mathrm{ICG}_n(\{1, 2, n/2\})$ also has a PST. Moreover, the condition $8 \mid n$ for the first two classes and $16 \mid n$ for the third class is necessary and sufficient for existence of PST.

Theorem 15 *Integral circulant graph $\mathrm{ICG}_n(\{1, n/4\})$ has PST if and only if $S_2(n) \geq 3$.*

Proof.

($\Leftarrow$:) The eigenvalues of $\mathrm{ICG}_n(\{1, n/4\})$ are given by $\lambda_j = c(j,n) + c(j,4)$.

Suppose that $j \in 2\mathbb{N}+1$. From (13) we have $c(j,4) = 0$. Moreover as $8 \mid t_{n,j}$ we also have $\mu(t_{n,j}) = c(j,n) = 0$ and thus $\lambda_j = 0$.

Now suppose that $j \in 4\mathbb{N}+2$. Similarly we conclude that $c(j,4) = -2$ for relation (13) and, as $4 \mid t_{n,j}$, that $\mu(t_{n,j}) = c(j,n) = 0$. Thus, in this case, there holds that $\lambda_j = -2$ for all $0 \leq j \leq n-1$ and $S_2(j) = 1$.

Finally let $j \in 4\mathbb{N}$. From (13) we have $c(j,4) = 2$. Denote $\alpha = S_2(n)$ and $\gamma = S_2(j)$. According to the assumptions we have $\alpha \geq 3$ and $\gamma \geq 2$. By direct computation we find

$$S_2(t_{n,j}) = S_2(n) - S_2(\gcd(j,n)) = \begin{cases} 0, & \alpha \leq \gamma \\ \alpha - \gamma, & \alpha > \gamma \end{cases}.$$

The last equation implies that term $\varphi(n)/\varphi(t_{n,j})$ is divisible by $2^{\min\{\alpha,\gamma\}}$ and therefore $4 \mid c(j,n)$ for $j \in 4\mathbb{N}$. Since $c(j,4) = 2$ we obtain $\lambda_j \in 4\mathbb{N}+2$.

According to the discussion above it holds that

$$S_2(\lambda_j) = \begin{cases} +\infty, & j \in 2\mathbb{N}+1 \\ 1, & j \in 2\mathbb{N} \end{cases}.$$

Using Lemma 6 we obtain that there is PST in $\mathrm{ICG}_n(\{1, n/4\})$ since $S_2(\lambda_{j+1} - \lambda_j) = 1$ for $0 \leq j \leq n-1$.

($\Rightarrow$:) We need to prove that there is no PST in graph $\mathrm{ICG}_n(\{1, n/4\})$ if $S_2(n) \leq 2$. The case $S_2(n) = 1$ was already considered in the previous section. The case $n = 4$ was already considered in [4] since it is a unitary Cayley graph. Therefore we suppose that $S_2(n) = 2$ and $n > 4$.

For $j \in 2\mathbb{N}+1$ we have, as in the previous theorem, that $4 \mid t_{n,j}$, $c(j,n) = 0$ and thus $\lambda_j = 0$.

According to Lemma 14 there exist integers j_1 and j_2 such that $c(j_1, n) \in 4\mathbb{N}+2$ and $c(j_2, n) \in 4\mathbb{N}$. Therefore $\lambda_{j_1} \in 4\mathbb{N}$ and $\lambda_{j_2} \in 4\mathbb{N}+2$. Since $\lambda_{j_1-1} = \lambda_{j_2-1} = 0$ ($j_1 - 1$ and $j_2 - 1$ are odd), it holds that $S_2(\lambda_{j_1} - \lambda_{j_1-1}) = 1$ and $S_2(\lambda_{j_2} - \lambda_{j_2-1}) \geq 2$, and according to Lemma 6 we conclude that there is no PST in $\mathrm{ICG}_n(\{1, n/4\})$.

$\square$

Theorem 16 *The integral circulant graph* $\mathrm{ICG}_n(\{1, n/2\})$ *has a PST if and only if* $S_2(n) \geq 3$.

Proof.

($\Leftarrow$:) The eigenvalues of $\mathrm{ICG}_n(\{1, n/2\})$ are given by $\lambda_j = c(j,n) + c(j,2)$. Let $\alpha = S_2(n)$. We prove that $4 \mid c(j,n)$ by distinguishing two cases.

Case 1. $S_2(j) \geq \alpha - 1$. Write $n = 2^\alpha m$ and $t_{n,j} = 2^\beta m'$ where $m, m' \in 2\mathbb{N}+1$. Since $S_2(t_{n,j}) = S_2(n) - S_2(\gcd(j,n))$ it holds that $\beta \leq 1$. Furthermore, we have

$$c(j,n) = \mu(t_{n,j}) \frac{\varphi(2^\alpha)\varphi(m)}{\varphi(2^\beta)\varphi(m')} = \mu(t_{n,j}) 2^{\alpha-1} \frac{\varphi(m)}{\varphi(m')}.$$

Since $\alpha \geq 3$, the last equation implies $4 \mid c(j,n)$.

Case 2. $S_2(j) < \alpha - 1$. Now $4 \mid t_{n,j}$. Since $t_{n,j}$ is not a square free integer, it holds that $\mu(t_{n,j}) = c(j,n) = 0$ and obviously $4 \mid c(j,n)$.

Now from $4 \mid c(j,n)$ and (12) we can conclude that $\lambda_j \in 4\mathbb{N} \pm 1$ and $S_2(\lambda_{j+1} - \lambda_j) = 1$. According to Lemma 6 there is a PST in $\mathrm{ICG}_n(\{1, n/2\})$.

($\Rightarrow$:) We need to prove that there is no PST in graph $\mathrm{ICG}_n(\{1, n/2\})$ if $S_2(n) \leq 2$. The case $S_2(n) = 1$ was already considered in the previous section. For $n = 4$ we directly check that $\mathrm{ICG}_4(\{1, 2\})$ does not have PST. Hence we suppose that $S_2(n) = 2$ and $n > 4$.

For $j \in 2\mathbb{N}+1$ it holds that $c(j,n) = 0$ and thus $\lambda_j = -1$ since $c(j,2) = -1$.

According to Lemma 14 there exist j_1 and j_2 such that $c(j_1, n) \in 4\mathbb{N}+2$ and $c(j_2, n) \in 4\mathbb{N}$. Since $c(j_1, 2) = c(j_2, 2) = 1$ it holds that $\lambda_{j_1} \in 4\mathbb{N}+1$ and $\lambda_{j_1} \in 4\mathbb{N}+3$. Moreover, since $j_1 - 1, j_2 - 1 \in 2\mathbb{N}+1$ we have $\lambda_{j_1-1} = \lambda_{j_2-1} = -1$, which further implies $S_2(\lambda_{j_1} - \lambda_{j_1-1}) = 1$ and $S_2(\lambda_{j_2} - \lambda_{j_2-1}) \geq 2$. According to Lemma 6 we conclude that there is no PST in $\mathrm{ICG}_n(\{1, n/2\})$. $\square$

Theorem 17 *The integral circulant graph* $\mathrm{ICG}_n(\{1,2,n/2\})$ *has a PST if and only if* $S_2(n) \geq 4$.

Proof.

($\Leftarrow$:) The eigenvalues of $\mathrm{ICG}_n(\{1,2,n/2\})$ are given by

$$\lambda_j = c(j,n) + c(j,n/2) + c(j,2).$$

Denote $\alpha = S_2(n)$. We will distinguish the following three cases.

Case 1. $S_2(j) \geq \alpha - 1$. Now $t_{n/2,j}$ is odd and similarly to the proof of the previous theorem, we can conclude that $2^{\alpha-1} \mid c(j,n/2)$. Since $S_2(t_{n,j}) = 2$ it holds that $2^{\alpha-1} \mid c(j,n)$.

Case 2. $S_2(j) = \alpha - 2$. Since $S_2(t_{n/2,j}) = 1$ there holds $2^{\alpha-2} \mid c(j,n/2)$. Moreover since $4 \mid t_{n,j}$ we can conclude that $c(j,n) = 0$.

Case 3. $S_2(j) < \alpha - 2$. Now both $t_{n,j}$ and $t_{n/2,j}$ are divisible by 4 and hence $c(j,n/2) = c(j,n)$.

We have seen that in all three cases $2^{\alpha-2} \mid c(j,n)+c(j,n/2)$. Since $\alpha \geq 4$ it holds that $4 \mid c(j,n)+c(j,n/2)$ and $\lambda_j \in 4\mathbb{N} \pm 1$. This further implies $\lambda_{j+1} - \lambda_j \in 4\mathbb{N}+2$ for every $j = 0,1,\ldots,n-2$. The existence of PST in $\mathrm{ICG}_n(\{1,2,n/2\})$ now follows directly from Lemma 6.

($\Rightarrow$:) Case $S_2(n) = 1$ was already considered in the previous section. For $n = 4$ we have that $\mathrm{ICG}_4(\{1,2\})$ does not have a PST. Hence we have to deal with the following two remaining cases.

Case 1. $S_2(n) = 2$ and $n > 4$. Suppose that $\mathrm{ICG}_n(\{1,2,n/2\})$ has a PST. Since $n/2$ is one of the divisors, λ_0 is odd. According to Corollary 7, all eigenvalues with odd indices must have the same parity.

Let $j \in 2\mathbb{N}+1$. Then $c(j,2) = -1$ and from $4 \mid t_{n,j}$ we obtain that $c(j,n) = 0$. Since $t_{n,2j} = t_{n/2,j}$ it holds that

$$c(j,n/2) = \mu(t_{n/2,j})\frac{\varphi(n/2)}{\varphi(t_{n/2,j})} = \mu(t_{n,2j})\frac{\varphi(n)}{2\varphi(t_{n,2j})} = \frac{1}{2}c(2j,n). \tag{15}$$

Since $S_2(n) = 2$, according to Lemma 14 there exist indices $j_1, j_2 \in 4\mathbb{N}+2$ such that $c(j_1,n) \in 4\mathbb{N}$ and $c(j_2,n) \in 4\mathbb{N}+2$. From (15) it hold that $c(j_1/2,n/2) \in 2\mathbb{N}$ and $c(j_2/2,n/2) \in 2\mathbb{N}+1$. Now $\lambda_{j_1/2} \in 2\mathbb{N}+1$ and $\lambda_{j_2/2} \in 2\mathbb{N}$, which is a contradiction since both $j_1/2$ and $j_2/2$ are odd.

Case 2. $S_2(n) = 3$. For $j \in 2\mathbb{N}+1$ we have $c(j,2) = -1$, $8 \mid t_{n,j}$, $4 \mid t_{n/2,j}$, which implies $c(j,n) = c(j,n/2) = 0$. Thus $\lambda_j = -1$.

For $j \in 4\mathbb{N}+2$, we have $c(j,2) = 1$, $4 \mid t_{n,j}$ and again $c(j,n) = 0$.

Since $S_2(n/2) = 2$, according to Lemma 14 there exist indices $j_1, j_2 \in 4\mathbb{N}+2$ such that $0 \leq j_1, j_2 \leq n/2-1$, $c(j_1,n/2) \in 4\mathbb{N}$ and $c(j_2,n/2) \in 4\mathbb{N}+2$. This implies that $\lambda_{j_1} \in 4\mathbb{N}+1$ and $\lambda_{j_2} \in 4\mathbb{N}-1$. Now $S_2(\lambda_{j_1} - \lambda_{j_1-1}) = 1$ and $S_2(\lambda_{j_2} - \lambda_{j_2-1}) \geq 2$. According to Lemma 6 there is no PST in $\mathrm{ICG}_n(\{1,2,n/2\})$. $\square$

5.2 Case $8 \nmid n$

The following theorems introduce two classes of ICG graphs having a PST in the case $S_2(n) = 2$.

Theorem 18 *Let* n *be a positive integer such that* $S_2(n) = 2$. *Then graph* $\mathrm{ICG}_n(\{1,2,4,n/4\})$ *has a PST.*

Proof. From relation (3), the eigenvalues of $\mathrm{ICG}_n(\{1,2,4,n/4\})$ are given by

$$\lambda_j = c(j,n) + c(j,n/2) + c(j,n/4) + c(j,4). \tag{16}$$

Now consider other summands in (16). First let us notice that since $n/4$ is an odd integer, $t_{n/4,j}$ must also be an odd integer. We distinguish two different cases depending on j.

Case 1. $2 \nmid j$. We prove that $\lambda_j = 0$. Since $\gcd(n,j) = \gcd(n/2,j) = \gcd(n/4,j)$ it holds that

$$t_{n/2,j} = \frac{n/2}{\gcd(n/2,j)} = 2\frac{n/4}{\gcd(n/4,j)} = 2t_{n/4,j}.$$

In the same way we can prove $t_{n,j} = 4t_{n/4,j}$. Now $c(j,n) = 0$ since $t_{n,j}$ is not a square-free integer. Furthermore, $\varphi(n/2) = \varphi(n/4)$ and $\varphi(t_{n/2,j}) = \varphi(t_{n/4,j})$ since $n/4$ and $t_{n/4,j}$ are odd integers. Also $\mu(t_{n/2,j}) = -\mu(t_{n/4,j})$ (if $t_{n/4,j}$ is not square-free, then both sides of the equation are equal to 0). Taking all these into account we finally obtain

$$c(n/2,j) = \mu(t_{n/2,j})\frac{\varphi(n/2)}{\varphi(t_{n/2,j})} = -\mu(t_{n/4,j})\frac{\varphi(n/4)}{\varphi(t_{n/4,j})} = -c(n/4,j) \tag{17}$$

Now by replacing the last equation in (16) and using (13) we finally obtain that $\lambda_j = 0$.

Case 2. $2 \mid j$. There holds $\gcd(n/2,j) = 2\gcd(n/4,j)$ which implies $t_{n/2,j} = t_{n/4,j}$. Also since $\varphi(n/2) = \varphi(n/4)$ there must hold $c(j,n/2) = c(j,n/4)$. Now we will distinguish two more cases depending on $S_2(j)$.

Case 2.1. $S_2(j) = 1$. We will prove that $\lambda_j = -2$. Since $\gcd(n,j) = \gcd(n/2,j)$ it holds that $t_{n,j} = 2t_{n/2,j}$. Since $t_{n/2,j} = t_{n/4,j}$ is odd, it holds that $\varphi(t_{n,j}) = \varphi(t_{n/2,j})$ and $\mu(t_{n,j}) = -\mu(t_{n/2,j})$. Furthermore using $\varphi(n) = 2\varphi(n/2)$ we obtain that

$$c(j,n) = \mu(t_{n,j})\frac{\varphi(n)}{\varphi(t_{n,j})} = -2\mu(t_{n/2,j})\frac{\varphi(n/2)}{\varphi(t_{n/2,j})} = -2c(j,n/2). \tag{18}$$

Now by replacing the last equation in (16), using (13) and $c(j,n/2) = c(j,n/4)$ we finally obtain that $\lambda_j = -2$.

Case 2.2. $S_2(j) \geq 2$. We will prove that $S_2(\lambda_j) = 1$. Similarly to the previous cases, it holds that $\gcd(n,j) = 4\gcd(n/4,j)$, $t_{n,j} = t_{n/4,j}$ and $c(j,n) = 2c(j,n/4)$. Now from (16) and (13) we obtain $\lambda_j = 4c(j,n/4)+2$, which implies that $S_2(\lambda_j) = 1$.

Summarizing all of the above considerations we have proved that

$$\lambda_j = \begin{cases} 0, & 2 \nmid j \\ -2, & S_2(j) = 1 \\ 4c(j,n/4)+2, & 4 \mid j \end{cases}. \tag{19}$$

From the last equation it is obvious that $S_2(\lambda_{j+1} - \lambda_j) = 1$ for every $j = 0,1,\ldots,n-2$. The existence of PST in $\mathrm{ICG}_n(\{1,2,4,n/4\})$ now follows from Lemma 6. $\square$

Theorem 19 *Let n be a positive integer such that $S_2(n) = 2$. Then graph $\mathrm{ICG}_n(\{1,2,4,n/2\})$ has a PST.*

Proof. From relation (3), the eigenvalues of $\mathrm{ICG}_n(\{1,2,4,n/2\})$ are given by

$$\lambda_j = c(j,n) + c(j,n/2) + c(j,n/4) + c(j,2). \tag{20}$$

The following relation for the eigenvalues λ_j can be proved in the same way as in the previous theorem

$$\lambda_j = \begin{cases} -1, & 2 \nmid j \\ 1, & S_2(j) = 1 \\ 4c(j,n/4)+1, & 4 \mid j \end{cases}, \tag{21}$$

The last equation yields $S_2(\lambda_{j+1} - \lambda_j) = 1$ for every $j = 0,1,\ldots,n-2$. The existence of PST in $\mathrm{ICG}_n(\{1,2,4,n/2\})$ now follows directly from Lemma 6. $\square$

Note that condition $S_2(n) = 2$ is not necessary for the existence of PST for both $\mathrm{ICG}_n(\{1,2,4,n/4\})$ and $\mathrm{ICG}_n(\{1,2,4,n/2\})$. For $n = 32$ it can be directly checked whether both graphs have a PST since $S_2(\lambda_{j+1} - \lambda_j) = 1$ for every $j = 0,1,\ldots,31$.

6 Some classes of $\mathrm{ICG}_n(D)$ having no PST for $4 \mid n$

In this section we find several classes of integral circulant graphs $\mathrm{ICG}_n(D)$ having no PST when $4 \mid n$. These classes have a special structure of the set of divisors D. More precisely, we will deal with the set of divisors D which has at most one even divisor. Hence we will assume until the end of the section that $4 \mid n$.

Theorem 20 *Let D contain only odd divisors. Then $\mathrm{ICG}_n(D)$ has no PST.*

Proof. Suppose that graph $\mathrm{ICG}_n(D)$ has a PST.

First let $S_2(n) \geq 3$. For each divisor $d \in D$ we have $8 \mid n/d$ and $4 \mid n/(2d)$. Hence $\mu(n/d) = \mu(n/(2d)) = 0$ which implies $c(1, n/d) = c(2, n/d) = 0$ and $\lambda_1 = \lambda_2 = 0$. This is in contradiction with Corollary 5.

Let $S_2(n) = 2$. We distinguish two cases.

Case 1. $n/4 \notin D$. Let $j \in 2\mathbb{N}+1$. For each divisor $d \in D$ we have $4 \mid t_{n/d,j}$. Consequently, $c(j, n/d) = 0$ which implies $\lambda_j = 0$.

Now let $j \in 4\mathbb{N}+2$ and denote $n_1 = n/2$. For an arbitrary $d \in D$ it holds that

$$t_{n/d,j} = \frac{n}{d \cdot \gcd(n/d, j)} = \frac{2n_1}{d \cdot 2\gcd(n_1/d, j/2)} = t_{n_1/d, j/2}.$$

Since $2 \mid n_1/d$ we have $\varphi(n/d) = 2\varphi(n_1/d)$. This implies that $c(j, n/d) = 2c(j/2, n_1/d)$ for every $d \in D$.

Consider the graph $\mathrm{ICG}_{n_1}(D)$ and denote its eigenvalues by λ'_j for $0 \leq j \leq n_1 - 1$. From the above discussion we conclude $\lambda_j = 2\lambda'_{j/2}$ for any $j \in 4\mathbb{N}+2$. Since $n_1 \in 2\mathbb{N}+1$ and all divisors in D are odd, according to Lemma 10 there exists $j_0 \in 4\mathbb{N}+2$ such that $\lambda'_{j_0/2} \in 2\mathbb{N}+1$. Consequently, $\lambda_{j_0} \in 4\mathbb{N}+2$.

Since $n/4 \notin D$ or, equivalently, $n_1/2 \notin D$ we have $\lambda'_0 \in 2\mathbb{N}$. Using

$$\lambda_0 = \sum_{d\in D} \varphi\left(\frac{n}{d}\right) = \sum_{d\in D} 2\varphi\left(\frac{n}{4d}\right) = 2\sum_{d\in D} \varphi\left(\frac{n_1}{2d}\right) = 2\sum_{d\in D} \varphi\left(\frac{n_1}{d}\right) = 2\lambda'_0$$

we conclude that $\lambda_0 \in 4\mathbb{N}$. Therefore $S_2(\lambda_0 - \lambda_1) \geq 2$ and $S_2(\lambda_{j_0} - \lambda_{j_0-1}) = 1$, which is in contradiction with Lemma 6.

Case 2. $n/4 \in D$. Let $D' = D \backslash \{n/4\}$. Denote by λ''_j the eigenvalues of the graph $\mathrm{ICG}_n(D')$. Thus we have the following relation between the eigenvalues of these graphs:

$$\lambda_j = \lambda''_j + c(j, 4).$$

For $j \in 2\mathbb{N}+1$, according to **Case 1** and relation (13) we have $\lambda_j = 0$. For the same reason it holds that $\lambda_0 \in 4\mathbb{N}+2$ and there exists $j_0 \in 4\mathbb{N}+2$ such that $\lambda_{j_0} \in 4\mathbb{N}$. Therefore $S_2(\lambda_0 - \lambda_1) = 1$ and $S_2(\lambda_{j_0} - \lambda_{j_0-1}) \geq 2$ which is in contradiction with Lemma 6. □

Theorem 21 *Let $S_2(n) = 2$ and D contains exactly one even divisor. Then $\mathrm{ICG}_n(D)$ has no PST.*

Proof. Suppose that graph $\mathrm{ICG}_n(D)$ has a PST. We distinguish two cases.

Case 1. $n/2 \in D$. Let $D' = D \backslash \{n/2\}$. Denote by λ'_j the eigenvalues of the graph $\mathrm{ICG}_n(D')$. Then

$$\lambda_j = \lambda'_j + c(j, 2).$$

Since $n/2 \in 4\mathbb{N}+2$, it is the only even divisor in D. Thus all divisors in D' are odd and we can proceed similarly as in the proof of Theorem 20. Hence there exists $j_0 \in 4\mathbb{N}+2$ such that $\lambda'_0 - \lambda'_1 \not\equiv \lambda'_{j_0} - \lambda'_{j_0-1} \pmod 4$. Moreover, using (12) we have $\lambda_0 - \lambda_1 = \lambda'_0 - \lambda'_1 + 2$ and $\lambda_{j_0} - \lambda_{j_1} = \lambda'_{j_0} - \lambda'_{j_1} + 2$. Thus $\lambda_0 - \lambda_1 \not\equiv \lambda_{j_0} - \lambda_{j_0-1} \pmod 4$, which is in contradiction with Lemma 6.

Case 2. $n/2 \notin D$. Then $\lambda_0 \in 2\mathbb{N}$. Let d_1 be the only even divisor in D.

Let $j \in 4\mathbb{N}+2$ and denote $n_1 = n/2$.

If $d_1 \in 4\mathbb{N}+2$ then

$$t_{n/d_1,j} = \frac{n}{d_1 \cdot \gcd(n/d_1, j)} = \frac{2n_1}{d_1 \cdot 2\gcd(n_1/d_1, j)} = \frac{n_1}{d_1 \cdot \gcd(n_1/d_1, j/2)} = t_{n_1/d_1, j/2}.$$

Moreover since $\varphi(n/d_1) = \varphi(n_1/d_1)$, by direct computation we find $c(j, n/d_1) = c(j/2, n_1/d_1)$. Since $n_1/d_1 \in 2\mathbb{N}+1$, Lemma 9 yields that there exists $j_0 \in 4\mathbb{N}+2$ such that $c(j_0, n/d_1) \in 2\mathbb{N}+1$.

Similarly, if we suppose $S_2(d_1) = 2$ then $c(j, n/d_1) = c(j/2, n_1/(d_1/2))$. Again since $n_1/(d_1/2) \in 2\mathbb{N}+1$, Lemma 9 yields that there exists $j_0 \in 4\mathbb{N}+2$ such that $c(j_0, n/d_1) \in 2\mathbb{N}+1$.

Furthermore, according to the proof of Theorem 20 we have $c(j, n/d) = 2c(j/2, n_1/d)$ for $j \in 4\mathbb{N}+2$ and odd $d \in D$. Thus, we finally obtain

$$\lambda_{j_0} = 2 \sum_{d \in D \setminus \{d_1\}} c(j_0/2, n_1/d) + c(j_0, n/d_1) \in 2\mathbb{N}+1.$$

Therefore, $\lambda_0 \in 2\mathbb{N}$ and $\lambda_{j_0} \in 2\mathbb{N}+1$, which is in contradiction with Corollary 7. $\square$

Theorem 22 *Let* $\mathrm{ICG}_n(D)$ *be an integral circulant graph such that* $S_2(n) \geq 3$. *Suppose that there exists an even divisor* $d_0 \in D$ *relatively prime with all other divisors* $d \in D \setminus \{d_0\}$. *Then* $\mathrm{ICG}_n(D)$ *has a PST if and only if* $D = \{1, n/2\}$ *or* $D = \{1, n/4\}$.

Proof.

($\Rightarrow$:) Suppose that $\mathrm{ICG}_n(D)$ has a PST.

Let $d \in D \setminus \{d_0\}$. First, note that d is odd since d and d_0 are relatively prime. Consequently, $S_2(n/d) \geq 3$ and $\mu(n/d) = \mu(n/(2d)) = 0$.

According to Proposition 2 it holds that

$$\lambda_1 = \mu(n/d_0), \quad \lambda_2 = \begin{cases} \mu(n/d_0), & n/d_0 \in 2\mathbb{N}+1 \\ \mu(n/(2d_0)), & n/d_0 \in 4\mathbb{N}+2 \\ 2\mu(n/(2d_0)), & n/d_0 \in 4\mathbb{N} \end{cases} \tag{22}$$

We will distinguish two cases according to d_0.

Case 1. Suppose that n/d_0 is a square-free integer. By (22) we obtain $\lambda_1 = \mu(n/d_0) \in 2\mathbb{N}+1$ and $\lambda_2 = c(2, n/d_0) \in \{\mu(n/d_0), \mu(n/2d_0)\} \subset 2\mathbb{N}+1$. Corollary 7 now yields that all eigenvalues are odd.

Moreover, since λ_0 is odd we have $n/2 \in D$. Since $n/2$ is an even integer, we conclude $d_0 = n/2$. As $\gcd(n/2, d) = 1$ for all divisors $d \in D \setminus \{n/2\}$ it then holds that d is not divisible by any prime factor of n. Hence, we have the only possible choice that d equals one and the set of divisors is $D = \{1, n/2\}$.

Case 2. Suppose that n/d_0 is not a square-free integer. If $S_2(n/d_0) \geq 3$ then $\mu(n/d_0) = \mu(n/(2d_0)) = 0$ and $\lambda_1 = \lambda_2 = 0$ according to (22). This is in contradiction with Corollary 5. Also, if there exists an odd prime p such that $S_p(n/d_0) \geq 2$, the same conclusion holds.

Therefore, for any odd prime $p \mid n/d_0$ it must hold $S_p(n/d_0) = 1$ and $S_2(n/d_0) = 2$ since n/d_0 is not square-free. In other words,

$$n/d_0 = 4p_1 p_2 \cdots p_l$$

where $n = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime factorization of n and $l \leq k$ (without loss of generality, we can suppose that the first l prime factors of n divides n/d_0). Furthermore, $\lambda_1 = 0$ and $\lambda_2 = 2(-1)^{l+1}$, according to (22) and the last conclusion. Now we distinguish two more cases.

Case 2.1. Suppose that $l \geq 1$. Thus $4 \mid \varphi(4)\varphi(p_1) \mid \varphi(n/d_0)$. Similarly, for each $d \in D \setminus \{d_0\}$ we have

$$\varphi\left(\frac{n}{d}\right) = \varphi\left(2^{\alpha_0} \frac{n}{2^{\alpha_0} d}\right) = 2^{\alpha_0 - 1} \varphi\left(\frac{n}{2^{\alpha_0} d}\right) \in 4\mathbb{N},$$

since $\alpha_0 = S_2(n) \geq 3$. Now

$$\lambda_0 = \varphi(n/d_0) + \sum_{d \in D \setminus \{d_0\}} \varphi(n/d) \in 4\mathbb{N},$$

wherefrom we obtain $S_2(\lambda_1 - \lambda_0) \geq 2$ and $S_2(\lambda_2 - \lambda_1) = 1$, which contradicts Lemma 6.

Case 2.2. Suppose that $l = 0$, i.e. $d_0 = n/4$. Since $d = 1$ is the only divisor of n, relatively prime with $n/4$, it must hold that $D = \{1, n/4\}$.

($\Leftarrow$:) Directly from Theorem 15 and Theorem 16. $\square$

Note that the graphs $\mathrm{ICG}_{30}(\{1,3,6\})$ and $\mathrm{ICG}_{30}(\{1,3,12\})$ have PST and the set of divisors contains exactly one even divisor. But they do not satisfy the condition of Theorem 16, since 6 and 12 are not relatively prime with 3.

Theorems 20, 21 and 22 yield to the following corollaries.

Corollary 23 *The integral circulant graph* $\mathrm{ICG}_n(D)$ *where* D *contains an even divisor which is relatively prime to all other divisors in* D*, has a PST if and only if* $S_2(n) \geq 3$ *and* $D = \{1, n/2\}$, $D = \{1, n/4\}$.

Corollary 24 *The integral circulant graph* $\mathrm{ICG}_n(D)$*, where* D *has exactly two divisors, has a PST if and only if* $S_2(n) \geq 3$ *and* $D = \{1, n/2\}$, $D = \{1, n/4\}$.

7 Conclusion

We proved that for $n \in 4\mathbb{N} + 2$ there is no integral circulant graph $\mathrm{ICG}_n(D)$ having a PST. Since the same result holds for $n \in 2\mathbb{N} + 1$ [4] we conclude that $4 \mid n$ is the necessary condition for the existence of PST in $\mathrm{ICG}_n(D)$. Moreover, we showed that, for $n \in 8\mathbb{N} + 4$ there are at least two and for $n \in 8\mathbb{N}$ at least three graphs having a PST.

The main result of this paper can be formulated as follows: For an arbitrary $n \in N$, there is an integral circulant graph of order n having a PST if and only if $4 \mid n$. On the other hand, we proved that $\mathrm{ICG}_n(D)$ does not have a PST if D contains only odd divisors.

Hence for every n divisible by 4 we can construct a quantum spin network with fixed nearest-neighbour couplings, based on an integral circulant graph having a PST.

From these results, the following question naturally arises: For a fixed order n, how many integral circulant graphs $\mathrm{ICG}_n(D)$ have a PST? Thus we made a computer program to answer the last question. Our program checks the existence of PST for every integral circulant graph $\mathrm{ICG}_n(D)$. Note that there are $2^{\tau(n)-1} - 1$ different integral circulant graphs of order n, where $\tau(n)$ is the number of divisors of n. The eigenvalues are computed using (3) and the existence of PST is checked using Lemma 6. The results are shown in the following tables:

n	4	8	12	16	20	24	28	32	36	40	44	48	52
# of ICG graphs with PST	1	2	2	4	2	10	2	8	4	10	2	44	2

n	56	60	64	68	72	76	80	84	88	92	96	100
# of ICG graphs with PST	10	10	16	2	44	2	44	10	10	2	184	4

n	104	108	112	116	120	124	128	132	136	140	144	148
# of ICG graphs with PST	10	8	44	2	218	2	32	10	10	10	400	2

n	152	156	160	164	168	172	176	180	184	188	192	196	200
# of ICG graphs with PST	10	10	184	2	218	2	44	44	10	2	752	4	44

It is worth mentioning that the maximum value of the *perfect quantum communication distance* (i.e. the distance between vertices where a perfect state transfer occurs) is equal to 2 for every $1 \leq n \leq 200$. One possible reason for such a low value of the perfect quantum communication distance is a small number of distinct prime divisors of graph order n (at most 3).

An improvement of the perfect quantum communication distance is done in [6] by considering fixed but different nearest-neighbor couplings. A similar approach used on circulant graphs (having a weighted adjacency matrix) might also enlarge the perfect quantum communication distance. Many recent papers propose an approach [10, 13, 14, 15].

We can see from the tables that for some values of n (for example $n = 96, 120, 144, 160, 168, 192, \ldots$) there are a lot of graphs having a PST, while for some other values, there are only 2 such graphs. For example, this is the case for $n = 4p$, where p is prime (see Appendix). Note that the number of graphs of order n with a PST is greater, for the greater value of $S_2(n)$. Hence it seems that the number of integral circulant graphs of order n having a PST can be expressed as some nice function of the parameters like $S_2(n)$, $\tau(n)$, etc. We leave this question for further research.

A Appendix: Number of ICG graphs having PST for $n = 4p$

According to results from tables in Section 7, we can conclude that the following theorem is valid:

Theorem 25 *Assume that $n = 4p$ where p is prime. There are exactly two ICG graphs of order n having PST. Those graphs are:*

- $\mathrm{ICG}_8(\{1,2\})$ *and* $\mathrm{ICG}_8(\{1,4\})$ *for* $p = 2$.
- $\mathrm{ICG}_n(\{1,2,4,p\})$ *and* $\mathrm{ICG}_n(\{1,2,4,2p\})$ *for* $p \geq 3$,

Proof.

According to Theorem 15 and Theorem 16 it holds that $\mathrm{ICG}_8(\{1,2\})$ and $\mathrm{ICG}_8(\{1,4\})$ have PST. Only remaining connected ICG different than unitary Cayley graph is $\mathrm{ICG}_8(\{1,2,4\})$. That graph is complete and holds $\lambda_1 = \lambda_2 = \ldots = \lambda_7 = -1$. Nonexistence of PST follows from Corollary 5.

Assume that $p \geq 3$. We distinguish several cases depending on the number of elements in D. Notice that $D_n = \{1, 2, 4, p, 2p\}$

Case 1. $|D| = 2$. Nonexistence of PST follows directly from Corollary 24.

Case 2. $|D| = 3$. Graphs $\mathrm{ICG}_n(\{1,2,p\})$, $\mathrm{ICG}_n(\{1,4,p\})$ and $\mathrm{ICG}_n(\{1,p,2p\})$ have no PST, according to Theorem 21. Also graph $\mathrm{ICG}_n(\{2,4,2p\}$ is not connected. We distinguish 6 more cases.

Case 2.1. $\mathrm{ICG}_n(\{1,2,2p\})$ and $\mathrm{ICG}_n(\{1,4,2p\})$. Since $n/2 = 2p \in D$, it holds that $\lambda_0 \in 2\mathbb{N}+1$ for both graphs. Also holds $\lambda_2 = 2\mu(2p) + \mu(p) + \mu(1) = 2$. For the first graph we have $\lambda_1 = \mu(4p) + \mu(2p) + \mu(2) = 0$ while for the second graph we have $\lambda_1 = \mu(4p) + \mu(p) + \mu(2) = -2$. In both cases, it holds that $S_2(\lambda_1 - \lambda_0) = 0$ and $S_2(\lambda_2 - \lambda_1) \geq 1$ which implies nonexistence of PST.

Case 2.2. $\mathrm{ICG}_n(\{2,p,2p\})$ and $\mathrm{ICG}_n(\{4,p,2p\})$. Since $n/2 = 2p \in D$, it holds that $\lambda_0 \in 2\mathbb{N}+1$ for both graphs. Also holds $\lambda_2 = \mu(p) + 2\mu(2) + \mu(1) = -2$. For the first graph we have $\lambda_1 = \mu(2p) + \mu(4) + \mu(2) = 0$ while for the second graph we have $\lambda_1 = \mu(p) + \mu(4) + \mu(2) = -2$. In both cases, it holds that $S_2(\lambda_1 - \lambda_0) = 0$ and $S_2(\lambda_2 - \lambda_1) \geq 1$ which implies nonexistence of PST.

Case 2.3. $\mathrm{ICG}_n(\{1,2,4\})$. Since $\lambda_1 = \mu(4p) + \mu(2p) + \mu(p) = 0$ and $\lambda_2 = 2\mu(2p) + \mu(p) + \mu(p) = 0$, nonexistence of PST follows from Corollary 5.

Case 2.4. $\mathrm{ICG}_n(\{2,4,p\})$. Since

$$t_{2p,3} = \frac{2p}{\gcd(2p,3)} = \begin{cases} 2, & p = 3 \\ 2p, & p > 3 \end{cases}, \quad t_{p,3} = \frac{p}{\gcd(p,3)} = \begin{cases} 1, & p = 3 \\ p, & p > 3 \end{cases}, \quad t_{4,3} = \frac{4}{\gcd(4,3)} = 4,$$

we have

$$c(3,2p) = \begin{cases} \mu(2)\frac{\varphi(2p)}{\varphi(2)} = -2, & p = 3 \\ \mu(2p)\frac{\varphi(2p)}{\varphi(2p)} = 1, & p > 3 \end{cases}, \quad c(3,p) = \begin{cases} \mu(1)\frac{\varphi(3)}{\varphi(1)} = 2, & p = 3 \\ \mu(p)\frac{\varphi(p)}{\varphi(p)} = -1, & p > 3 \end{cases}, \quad c(3,4) = 0,$$

which implies $\lambda_3 = c(3,2p) + c(3,p) + c(3,4) = 0$.

On the other side, it holds that

$$t_{2p,4} = \frac{2p}{\gcd(2p,4)} = p, \quad t_{p,4} = \frac{p}{\gcd(p,4)} = p, \quad t_{4,4} = \frac{4}{\gcd(4,4)} = 1,$$

which implies

$$c(4,2p)=\mu(p)\frac{\varphi(2p)}{\varphi(p)}=1,\quad c(4,p)=\mu(p)\frac{\varphi(p)}{\varphi(p)}=-1,\quad c(4,4)=\mu(1)\frac{\varphi(4)}{\varphi(1)}=2$$

and $\lambda_4=c(4,2p)+c(4,p)+c(4,4)=0$. Since $\lambda_3=\lambda_4=0$, nonexistence of PST follows from Corollary 5.

Case 3. $|D|=4$. We distinguish 3 more cases.

Case 3.1. Graphs $\mathrm{ICG}_n(\{1,2,4,p\}$ and $\mathrm{ICG}_n(\{1,2,4,2p\})$ have PST, according to Theorem 18 and Theorem 19.

Case 3.2. $\mathrm{ICG}_n(\{1,2,p,2p\}$ and $\mathrm{ICG}_n(\{1,4,p,2p\})$. Since $n/2=2p\in D$, it holds that $\lambda_0\in 2\mathbb{N}+1$ for both graphs. Also holds $\lambda_2=2\mu(2p)+\mu(p)+2\mu(2)+\mu(1)=0$. For the first graph we have $\lambda_1=\mu(4p)+\mu(2p)+\mu(4)+\mu(2)=0$ while for the second graph we have $\lambda_1=\mu(4p)+\mu(p)+\mu(4)+\mu(2)=-2$. In both cases, it holds that $S_2(\lambda_1-\lambda_0)=0$ and $S_2(\lambda_2-\lambda_1)\geq 1$ which implies nonexistence of PST.

Case 3.3. $\mathrm{ICG}_n(\{2,4,p,2p\}$. It holds that $\lambda_0=\varphi(2p)+\varphi(p)+\varphi(4)+\varphi(2)=2p+1\in 4\mathbb{N}+3$, $\lambda_1=\mu(2p)+\mu(p)+\mu(4)+\mu(2)=-1$ and $\lambda_2=\mu(p)+\mu(p)+2\mu(2)+\mu(1)=-3$. Now $S_2(\lambda_1-\lambda_0)\geq 2$ and $S_2(\lambda_2-\lambda_1)=1$ implies nonexistence of PST.

Case 4. $|D|=5$. Since $D=D_n$ we have that $\mathrm{ICG}_n(D)$ is complete graph and hence does not have PST.

$\square$

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