Some classes of integral circulant graphs either allowing or not allowing perfect state transfer*

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Abstract

The existence of perfect state transfer in quantum spin networks based on integral circulant graphs has been considered recently by Saxena, Severini and Shparlinski. Motivated by the mentioned work, Bašić, Petković and Stevanović give the simple condition for the characterization of integral circulant graphs allowing the perfect state transfer in terms of its eigenvalues. They stated that integral circulant graphs with minimal vertices allowing perfect state transfer, other than unitary Cayley graphs, are $\mathrm{ICG}_8(\{1,2\})$ and $\mathrm{ICG}_8(\{1,4\})$. Moreover, it is also conjectured that two classes of integral circulant graphs $\mathrm{ICG}_n(\{1,n/4\})$ and $\mathrm{ICG}_n(\{1,n/2\})$ allow PST where $n \in 8\mathbb{N}$. These conjectures are confirmed in this paper. Moreover, it is shown that there are no integral circulant graphs allowing perfect state transfer in the class of graphs where the number of vertices is square-free integer.

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1 Introduction

Integral graphs are extensively studied in the literature. Harary and Schwenk first studied integral graphs in the paper [6]. A graph is called an *integral* if all eigenvalues are integers. Also there is vast research on the class of circulant graphs. Graph is called *circulant* if it is Cayley graph on circulant group, i.e. its adjacency matrix is circulant. These graphs have important application as a class of interconnection networks in parallel and distributed computing. On the other side, the first characterization of graphs belonging to both classes of integral and circulant graphs is given by So in [12]. These graphs are called *integral circulant graphs.*

Moreover, integral circulant graphs are the generalization of well-known class of unitary Cayley graphs. Various properties of unitary Cayley graphs were investigated in some recent papers. For example, Klotz and Sander [9] determined the diameter, clique number, chromatic number and eigenvalues of unitary Cayley graphs. The problem of longest induced cycle in unitary Cayley graphs is studied in the paper of Fuchs [5] and also in the paper of Berrizbeitia and Giudici [3]. On the other side, there is almost no research on integral circulant graphs. Some results about clique number integral circulant graphs with exactly one and two divisors are recently obtained by Bašić and Ilić [1].

In the recent work of Saxena, Severini and Shparlinski it is stated that the integral circulant graphs are potential candidates for modeling quantum spin networks that might enable the perfect state transfer between antipodal sites in a network. Discrete quantum networks allowing perfect state transfer are recently considered in the paper [4]. More results in this topic are given by Bašić, Petković and Stevanović in paper [2]. Simple and general characterization of the existence of perfect state transfer in integral circulant graphs, in terms of its eigenvalues is given in that paper.

It is proven that integral circulant graphs with odd number of vertices do not allow perfect state transfer. Also, it is given the characterization of unitary Cayley graphs allowing perfect state transfer.

However, the general problem of characterizing integral circulant graphs having perfect state transfer is still open. In this paper we give the partial answer to this problem. It is shown that graphs whose number of vertices is square-free integer, do not allow perfect state transfer. Moreover, we found two general classes of integral circulant graphs allowing perfect state transfer. These are the first known classes of integral circulant graphs whose all members allow perfect state transfer.

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2 Integral circulant graphs

A *circulant graph* $G(n;S)$ is graph on vertices $\mathbb{Z}_n = \{0,1,\ldots,n-1\}$ such that each vertex i is adjacent to vertices $i+s$ for all $s \in S$. A set S is called a *symbol* of graph $G(n;S)$. Note that the degree of graph $G(n;S)$ is $\#S$. Graph is *integral* if all its eigenvalues are integers. Wasin So has characterized integral circulant graphs [12] by the following theorem

Theorem 1 **[12]** *A circulant graph $G(n;S)$ is integral if and only if*

$$S = \bigcup_{d \in D} G_n(d),$$

for some set of divisors $D \subseteq D_n$. Here $G_n(d) = \{k \ : \ \gcd(k,n) = d,\ 1 \le k \le n-1\}$, and D_n is the set of all divisors of n less than n.

Therefore an integral circulant graph $G(n;S)$ is defined by its order n and the set of divisors D. Such graphs are also known as *gcd-graphs* (see for example [9]). An integral circulant graph with n vertices defined with the set of divisors $D \subseteq D_n$ will be denoted by $\mathrm{ICG}_n(D)$. From Theorem 1 we have that the degree of an integral circulant graph is equal to $\deg \mathrm{ICG}_n(D) = \sum_{d \in D} \varphi(n/d)$. Here $\varphi(n)$ denotes Euler-phi function [7]. The next theorem concerns the connectivity of an integral circulant graphs.

The eigenvalues and eigenvectors of $\mathrm{ICG}_n(D)$ are given by [10]

$$\lambda_j = \sum_{s \in S} \omega_n^{js}, \quad v_j = [1 \ \omega_n^s \ \omega_n^{2s} \cdots \omega_n^{(n-1)s}],$$

where $\omega_n = e^{i\frac{2\pi}{n}}$ is the n-th root of unity. Denote by $c(j,n)$ the following expression

$$c(j,n) = \mu(t_{n,j}) \frac{\varphi(n)}{\varphi(t_{n,j})}, \quad t_{n,j} = \frac{n}{\gcd(n,j)}, \tag{1}$$

where μ is Möbius function. Expression $c(j,n)$ is known as the *Ramanujan function* [7]. Eigenvalues λ_j can be expressed in terms of Ramanujan function as follows ([9], Theorem 16)

$$\lambda_j = \sum_{d \in D} c(j, n/d). \tag{2}$$

Let us observe the following properties of the Ramanujan function. These basic properties will be used in the rest of the paper.

Proposition 2 *For any positive integers n, j and d such that $d \mid n$, holds*

$$c(0,n) = \varphi(n), \tag{3}$$

$$c(1,n) = \mu(n), \tag{4}$$

$$c(2,n) = \begin{cases} \mu(n), & n \in 2\mathbb{N}+1 \\ \mu(n/2), & n \in 4\mathbb{N}+2 \\ 2\mu(n/2), & n \in 4\mathbb{N} \end{cases} \tag{5}$$

$$c(n/2, n/d) = \begin{cases} \varphi(n/d), & d \in 2\mathbb{N} \\ -\varphi(n/d), & d \in 2\mathbb{N}+1 \end{cases} \tag{6}$$

Proof. Directly by using relation (1). $\square$

3 Perfect state transfer

For a given integral circulant graph $\mathrm{ICG}_n(D)$ we say that there is a *perfect state transfer* (PST) between vertices a and b if there is a positive real number t such that

$$|\langle a|e^{iAt}|b\rangle| = \left| \frac{1}{n} \sum_{l=0}^{n-1} e^{i\lambda_l t} \omega_n^{l(a-b)} \right| = 1. \tag{7}$$

We restate some results proved in our paper [2].

Theorem 3 **[2]** *There exists PST in graph* $\mathrm{ICG}_n(D)$ *between vertices* a *and* 0 *if and only if there are integers* p *and* q *such that* $\gcd(p,q)=1$ *and*

$$\frac{p}{q}(\lambda_{j+1}-\lambda_j)+\frac{a}{n}\in\mathbb{Z}, \tag{8}$$

for all $j=0,\ldots,n-2$.

Theorem 4 **[2]** *There is no PST in* $\mathrm{ICG}_n(D)$ *if* n/d *is odd for every* $d\in D$. *For* n *even, if there exists PST in* $\mathrm{ICG}_n(D)$ *between vertices* a *and* 0 *then* $a=n/2$.

According to the last theorem, PST is possible in $\mathrm{ICG}_n(D)$ just for even n and between vertices $a=n/2$ and 0 (i.e. between b and $n/2+b$ as mentioned in [10]). Therefore, in the rest of the paper we assume that n is even and $a=n/2$.

For a given prime number p and integer $n\in\mathbb{N}_0$, denote by $S_p(n)$ the maximal number α such that $p^\alpha\mid n$ if $n\in\mathbb{N}$, and $S_p(0)=+\infty$ for an arbitrary prime number p.

Lemma 5 **[2]** *There exists PST in* $\mathrm{ICG}_n(D)$, *if and only if there exists number* $m\in\mathbb{N}_0$ *such that the following holds for all* $j=0,1,\ldots,n-2$

$$S_2(\lambda_{j+1}-\lambda_j)=m. \tag{9}$$

Next corollary follows directly from Lemma 5.

Corollary 6 *Let* $\mathrm{ICG}_n(D)$ *has PST. One of the following two statements must hold*

1. *All eigenvalues* λ_j *have the same parity.*
2. *All eigenvalues* λ_j *with odd index* j *have the same parity and same holds for an even index* j *(i.e.* λ_j *are alternatively odd and even).*

We end this section with the following result concerning unitary Cayley graphs.

Theorem 7 **[2]** *The only unitary Cayley graphs that have PST are* K_2 *and* C_4.

Therefore in the rest of the paper we assume that set D has at least two divisors, i.e. $|D|\geq 2$.

4 PST on integral circulant graphs whose order is an even square-free integer

Let n is an even square-free integer, i.e. let $n=p_1p_2\cdots p_k$ where p_i are distinct primes for $i=1,\ldots,k$ and $p_1=2$. Main result of this section is the following theorem.

Theorem 8 *There is no PST in graph* $\mathrm{ICG}_n(D)$ *if* n *is even square-free integer.*

First observe that $\varphi(n)=\varphi(t_{n,j})\varphi(n/t_{n,j})$, due to the multiplicity of φ and the fact that n is square-free integer. Relation remains true if n is exchanged with n/d for arbitrary $d\in D$. Expressions (1) and (2), for Ramanujan sum and eigenvalues of $\mathrm{ICG}_n(D)$ respectively, reduce to

$$c(j,n)=\mu(t_{n,j})\varphi(n/t_{n,j})=\mu(t_{n,j})\varphi(\gcd(j,n)),\quad \lambda_j=\sum_{d\in D}\mu(t_{n/d,j})\varphi(\gcd(j,n/d)). \tag{10}$$

Proposition 9 *Eigenvalues* λ_{2p} *and* λ_p *have the same parity for arbitrary prime divisor* $p>2$ *of* n.

Proof. Observe that

$$\gcd(p,n/d)=\left\{\begin{array}{ll} p, & p\nmid d\\ 1, & p\mid d\end{array}\right.,\quad \gcd(2p,n/d)=\left\{\begin{array}{ll} 2p, & p\nmid d,\ 2\nmid d\\ p, & p\nmid d,\ 2\mid d\\ 2, & p\mid d,\ 2\nmid d\\ 1, & p\mid d,\ 2\mid d\end{array}\right.,$$

which directly yields to

$$\varphi(\gcd(p,n/d))=\varphi(\gcd(2p,n/d))=\left\{\begin{array}{ll} p-1, & p\nmid d\\ 1, & p\mid d\end{array}\right..$$

Now the conclusion follows from (10) using the fact that Moebius function takes values 1 and -1 on square-free arguments. $\square$

As a direct corollary of Proposition 9 and Corollary 6 holds that if $\mathrm{ICG}_n(D)$ has PST, all eigenvalues λ_j have the same parity. Next we consider two cases depending on the condition that $n/2 \in D$.

Case 1. Let $n/2 \notin D$. Then $\lambda_0 = \sum_{d\in D} \varphi(n/d)$ is even ($n/d > 2$ for any $d \in D$) and therefore if $\mathrm{ICG}_n(D)$ has PST, all eigenvalues λ_j are even.

Next we establish some properties of integral circulant graphs $\mathrm{ICG}_n(D)$ whose all eigenvalues are even.

Lemma 10 *If all eigenvalues of* $\mathrm{ICG}_n(D)$ *are even, then for arbitrary odd divisor* j *of* n *holds* $\#\{d \in D \ : \ j \mid d\} \in 2\mathbb{N}$.

Proof. Observe that for arbitrary odd prime divisors of n, $p_{i_1}, \ldots, p_{i_l}$ and arbitrary $d \in D$ holds

$$\varphi(\gcd(p_{i_1}\cdots p_{i_l}, n/d)) = \prod_{p_{i_k}\nmid d} (p_{i_k} - 1).$$

Last expression is even except in the case when all p_{i_k} divide d. In such case, $\varphi(\gcd(p_{i_1}\cdots p_{i_l}, n/d)) = 1$. By setting $j = p_{i_1}p_{i_2}\cdots p_{i_l}$ and using the fact that λ_j is even we conclude that the number of divisors $d \in D$ such that $j \mid d$, is even. In other words, for arbitrary divisor j of n, holds

$$\#\{d \in D \ : \ j \mid d\} \in 2\mathbb{N}. \tag{11}$$

□

The following lemma introduces the pairing between odd and even divisors in D. In other words, it states that d and $2d$ are either both in D or both not in D.

Lemma 11 *If all eigenvalues* λ_j *are even then for every odd* $d \in D$ *holds* $2d \in D$ *and for every even* $d \in D$ *holds* $d/2 \in D$.

Proof. Suppose that statement of lemma is not true. Let d_0 be maximal odd divisor of n such that exactly one of the numbers d_0 and $2d_0$ does not belong to D. Let us count the divisors $d \in D$ such that $d_0 \mid d$. Suppose that $d > 2d_0$. If d is odd and $d_0 \mid d$, then also $2d \in D$ (since d_0 is maximal) and $d_0 \mid 2d$. On the other side, if $d \in D$ is even and $d_0 \mid d$, then also $d_0 \mid d/2$ and $d/2 \in D$ (since d_0 is maximal).

This proves that there are an even number of divisors $d \in D$ such that $d > 2d_0$ and $d_0 \mid d$. There is also one more divisor in D (d_0 or $2d_0$) which does not satisfy $d > 2d_0$. Therefore total number of divisors $d \in D$ such that $d_0 \mid d$ is odd, which contradicts (11). □

Lemma 12 *If all eigenvalues of* $\mathrm{ICG}_n(D)$ *are even, the following relations hold* $S_2(\lambda_0) > 1$, $\lambda_{2j+1} = 0$ *for all* $j = 0, 1, \ldots, n/2 - 1$ *and* $S_2(\lambda_k) = 1$ *for some even* $k \in \{0, 1, \ldots, n-1\}$.

Proof. Note that for odd j holds $\gcd(j, n/(2d)) = \gcd(j, n/d)$ and $\mu(t_{n/(2d),j}) = -\mu(t_{n/d,j})$. Using (10) we obtain that $\lambda_j = 0$ for j odd. For even j holds $2\gcd(j, n/(2d)) = \gcd(j, n/d)$ and $\mu(t_{n/(2d),j}) = \mu(t_{n/d,j})$. Again using (10) we obtain

$$\lambda_j = 2 \sum_{d\in D\cap 2\mathbb{N}+1} \mu(t_{n/d,j})\varphi(\gcd(j, n/d)),$$

for even j.

Consider now the new integral circulant graph $\mathrm{ICG}_{n_1}(D_1)$ where $n_1 = n/2$ and $D_1 = D \cap 2\mathbb{N}+1$. Denote its eigenvalues by λ'_j for $j = 0, \ldots, n_1 - 1$. It holds

$$\lambda'_j = \sum_{d\in D_1} \mu(t_{n_1,j})\varphi(\gcd(j, n_1/d)) = \sum_{d\in D_1} \mu(t_{2n_1,2j})\varphi(\gcd(2j, 2n_1/d)) = \lambda_{2j}/2.$$

Observe that since $n_1/2 \notin D$ (n_1 is odd) holds $\lambda_0 \in 2\mathbb{N}$. Let $d'_{max} = \max D_1$. Since all divisors from D_1 are odd, Lemma 11 yields that there is at least one odd eigenvalue. Since $\#\{d' \in D_1 \ : \ d'_{max} \mid d'\} = \#\{d'_{max}\} = 1$, according to the proof of Lemma 10 holds that $\lambda'_{d'_{max}}$ is odd. Therefore

$$S_2(\lambda_{2d'_{max}}) = S_2(\lambda'_{d'_{max}}) + 1 = 1, \quad S_2(\lambda_0) = S_2(\lambda'_0) + 1 > 1.$$

By taking $k = 2d'_{max}$ we prove both statements in the lemma. □

Now we are ready to prove the main theorem in this case.

Theorem 13 *If* $n/2 \notin D$ *then graph* $\mathrm{ICG}_n(D)$ *does not have PST.*

Proof. If $\mathrm{ICG}_n(D)$ allows PST and $\lambda_0 \in 2\mathbb{N}$, all eigenvalues λ_j must be even for $j = 0, 1, \ldots, n-1$. Moreover it must hold $S_2(\lambda_{2j} - \lambda_{2j-1}) = S_2(\lambda_{2j}) = const$ for all $j = 0, 1, \ldots, n/2 - 1$ (according to the Lemma 5). This is the contradiction with Lemma 12. □

Case 2. Now let $n/2 \in D$. Since $\lambda_0 = \sum_{d \in D} \varphi(n/d)$ is odd, if there exists PST in $\mathrm{ICG}_n(D)$ then all eigenvalues λ_j must be odd.

Let $D_1 = D \setminus \{n/2\}$. Denote by λ'_j eigenvalues of an integral circulant graph $\mathrm{ICG}_n(D_1)$. Since

$$c(j,2) = \begin{cases} 1, & j \in 2\mathbb{N} \\ -1, & j \in 2\mathbb{N}+1 \end{cases},$$

there holds

$$\lambda_j = \begin{cases} \lambda'_j + 1, & j \in 2\mathbb{N} \\ \lambda'_j - 1, & j \in 2\mathbb{N}+1 \end{cases}. \tag{12}$$

Theorem 14 *There is no PST in* $\mathrm{ICG}_n(D)$ *if* $n/2 \in D$.

Proof. Graph $\mathrm{ICG}_n(D_1)$ has all even eigenvalues according to (12) and Lemma 12 yields $S_2(\lambda'_0) > 1$, $\lambda'_{2j+1} = 0$ for all $j = 0, 1 \ldots, n/2 - 1$ and $S_2(\lambda'_k) = 1$ for some even k. Then

$$S_2(\lambda_1 - \lambda_0) = S_2(\lambda'_1 - \lambda'_0 - 2) = S_2(-\lambda'_0 - 2) = 1,$$

and also

$$S_2(\lambda_k - \lambda_{k-1}) = S_2(\lambda'_k - \lambda'_{k-1} + 2) = S_2(\lambda'_k + 2) > 1.$$

This is contradiction with Lemma 5. □

5 Non-unitary integral circulant graph having PST with minimal number of vertices

Proposition 15 *Minimal number of vertices of non-unitary integral circulant graph allowing PST is* $n = 8$.

Proof. Let n be minimal number of vertices such that there exists graph $\mathrm{ICG}_n(D)$ having PST. We will prove that $n \geq 8$. According to the Theorem 4 holds $n \neq 3, 5, 7$. From Theorem 8 we have $n \neq 6$ since it is square-free integer.

Rest we need to prove that for $n = 2$ and $n = 4$ there are no non-unitary integral circulant graphs having PST. For $n = 2$ there is only one graph $\mathrm{ICG}_2(\{1\}) = K_2$ which is unitary. For $n = 4$ it is only $\mathrm{ICG}_4(\{1,2\})$ (since $\mathrm{ICG}_4(\{2\}) = 2K_2$). Its eigenvalues are $\lambda_0 = 3$ and $\lambda_1 = \lambda_2 = \lambda_3 = -1$ and according to Lemma 5 there is no PST.

For $n = 8$, graphs $\mathrm{ICG}_8(\{1,4\})$ and $\mathrm{ICG}_8(\{1,2\})$ have PST. Their spectra are $\{5, -1, 1, -1, -3, -1, 1, -1\}$ and $\{6, 0, -2, 0, -2, 0, -2, 0\}$ respectively. Value $S_2(\lambda_{j+1} - \lambda_j)$ is equal 1 in both cases for every $j = 0, 1, \ldots, 7$ and according to Lemma 5 there is PST. □

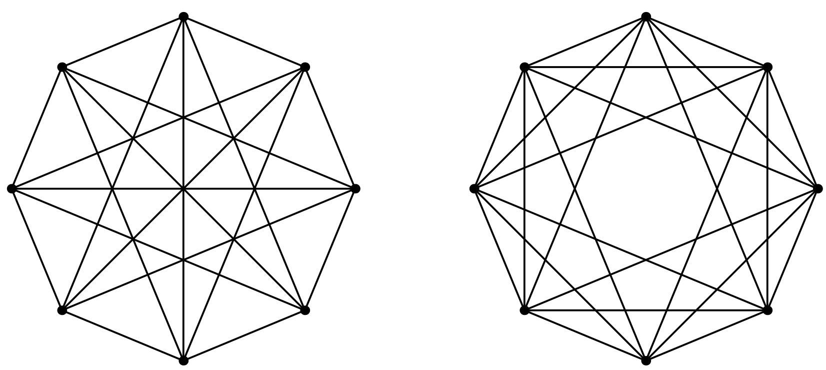

Figure 1: Non-unitary integral circulant graphs with minimal number of vertices ($\mathrm{ICG}_8(\{1,4\})$ on the left and $\mathrm{ICG}_8(\{1,2\})$ on the right) having PST

6 Two classes of integral circulant graphs which have a PST

Consider the following two classes of integral circulant graph with two divisors: $\mathrm{ICG}_n(\{1, n/2\})$ and $\mathrm{ICG}_n(\{1, n/4\})$. These classes are mentioned in [2] as the possible candidates for existence of PST. It is stated that both classes allow PST if $8 \mid n$. In this section we give the proof of this result.

Recall that these classes are the first complete classes of integral circulant graphs allowing PST.

Theorem 16 *Integral circulant graph* $\mathrm{ICG}_n(\{1, n/2\})$ *where* $S_2(n) \geq 3$ *has PST.*

Proof. Eigenvalues of $\mathrm{ICG}_n(\{1, n/2\})$ are given by $\lambda_j = c(j, n) + c(j, 2)$. It can be easily proven that

$$c(j,2) = \begin{cases} 1, & 2 \mid j \\ -1, & 2 \nmid j \end{cases}. \tag{13}$$

Let $\alpha = S_2(n)$.

Suppose that $S_2(j) \geq \alpha - 1$. Since $t_{n,j} = n/\gcd(n,j)$ it yields $4 \nmid t_{n,j}$. Write $n = 2^\alpha m$ and $t_{n,j} = 2^\beta m'$ where $m, m' \in 2\mathbb{N}+1$. We have already proven that $\beta \leq 1$. Then holds

$$c(j,n) = \mu(t_{n,j}) \frac{\varphi(2^\alpha)\varphi(m)}{\varphi(2^\beta)\varphi(m')} = \mu(t_{n,j}) 2^{\alpha-1} \frac{\varphi(m)}{\varphi(m')}.$$

Since $\alpha \geq 3$ last equation implies $4 \mid c(j,n)$.

Otherwise, if $S_2(j) < \alpha - 1$ then $4 \mid t_{n,j}$. Since $t_{n,j}$ it is not square-free integer there holds $\mu(t_{n,j}) = c(j,n) = 0$ and obviously $4 \mid c(j,n)$.

Now from $4 \mid c(j,n)$ and (13) we conclude that $\lambda_j \in 4\mathbb{N} \pm 1$ and $S_2(\lambda_{j+1} - \lambda_j) = 1$. According to the Lemma 5 there is PST in $\mathrm{ICG}_n(\{1, n/2\})$. □

Theorem 17 *Integral circulant graph* $\mathrm{ICG}_n(\{1, n/4\})$ *where* $S_2(n) \geq 3$ *has PST.*

Proof. Eigenvalues of $\mathrm{ICG}_n(\{1, n/4\})$ are given by $\lambda_j = c(j, n) + c(j, 4)$.

By direct computation we find

$$t_{4,j} = \frac{4}{\gcd(4,j)} = \begin{cases} 4, & 2 \nmid j \\ 2, & S_2(j) = 1 \\ 1, & 4 \mid j \end{cases}, \quad c(j,4) = \begin{cases} 0, & 2 \nmid j \\ -2, & S_2(j) = 1 \\ 2, & 4 \mid j \end{cases}. \tag{14}$$

Let $j \in 2\mathbb{N}+1$. From (14) we have $c(j,4) = 0$. Moreover as $8 \mid t_{n,j}$ we also have $\mu(t_{n,j}) = c(j,n) = 0$ and thus $\lambda_j = 0$.

Now let $j \in 4\mathbb{N}+2$. Similarly we conclude $c(j,4) = -2$ for relation (14) and as $4 \mid t_{n,j}$ we have $\mu(t_{n,j}) = c(j,n) = 0$. Thus in this case, there holds $\lambda_j = -2$ for all $0 \leq j \leq n-1$ and $S_2(j) = 1$.

Finally let $j \in 4\mathbb{N}$. From (14) we have $c(j,4) = 2$. Denote $\alpha = S_2(n)$ and $\gamma = S_2(j)$. According to assumptions we have $\alpha \geq 3$ and $\gamma \geq 2$. By the definition of the term $t_{n,j}$ there holds

$$S_2(t_{n,j}) = \begin{cases} 0, & \alpha \leq \gamma \\ \alpha - \gamma, & \alpha > \gamma \end{cases}.$$

Last equation yields that the term $\varphi(n)/\varphi(t_{n,j})$ is divisible by $2^{\min\{\alpha,\gamma\}}$ and therefore $4 \mid c(j,n)$ for $j \in 4\mathbb{N}$. Since $c(j,4) = 2$ we obtain $\lambda_j \in 4\mathbb{N}+2$.

According to above discussion there holds

$$S_2(\lambda_j) = \begin{cases} +\infty, & j \in 2\mathbb{N}+1 \\ 1, & j \in 2\mathbb{N} \end{cases}.$$

Using Lemma 5 we obtain that there is PST in $\mathrm{ICG}_n(\{1, n/4\})$ since $S_2(\lambda_{j+1} - \lambda_j) = 1$ for $0 \leq j \leq n-1$. □

7 Conclusion

This paper provides the further study on the PST existence problem in integral circulant graphs. It is proven that if n is square-free integer, there is no integral circulant graph allowing PST. This result is the continuation of the research performed in [2]. As the corollary, we obtained that integral circulant graphs with minimal vertices allowing PST, other than unitary Cayley graphs, are $\mathrm{ICG}_8(\{1,2\})$ and $\mathrm{ICG}_8(\{1,4\})$. Moreover we made a step forward and proved that two classes of integral circulant graphs $\mathrm{ICG}_n(\{1,n/4\})$ and $\mathrm{ICG}_n(\{1,n/2\})$ allow PST when $n \in 8\mathbb{N}$. This is the first result in this topic concerning the complete classes of integral circulant graphs allowing PST.

The following question now comes naturally: Are there integral circulant graphs with non square-free number of vertices allowing PST, such that $n \notin 8\mathbb{N}$? In particular, are there any such graphs when set of divisor D contains exactly two elements. To illustrate these questions we can compute the eigenvalues of graphs $\mathrm{ICG}_{12}(\{1,6\})$ and $\mathrm{ICG}_{12}(\{1,3\})$ and conclude that they do not allow PST. Therefore, condition $S_2(n) \geq 3$ in Theorem 16 and Theorem 17 cannot be weakened. On the other side, graphs $\mathrm{ICG}_{12}(\{1,2,4,6\})$ and $\mathrm{ICG}_{12}(\{1,2,3,4\})$ allow PST. Notice that integral circulant graphs $\mathrm{ICG}_n(D)$ where $9 \leq n \leq 11$ do not allow PST since n is odd or square-free number.

Last example also suggests the following reverse question: For which number n there exists integral circulant graph $\mathrm{ICG}_n(D)$ allowing PST where $|D| = 2$? And in addition, what is the least number of divisors in D for an integral circulant graph allowing PST with a given number of vertices n?

All these questions are the special case of the general problem: Characterize all integral circulant graphs $\mathrm{ICG}_n\{D\}$ allowing PST according to number of vertices n with the least number of divisors in D. Since there are different complete classes of integral circulant graphs, either allowing or not allowing PST, it follows that the mentioned characterization problem is very hard in general case. We leave this for the further study.

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