Perfect state transfer in integral circulant graphs *

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Abstract

The existence of perfect state transfer in quantum spin networks based on integral circulant graphs has been considered recently by Saxena, Severini and Shparlinski. We give the simple condition for characterizing integral circulant graphs allowing the perfect state transfer in terms of its eigenvalues. Using that we complete the proof of results stated by Saxena, Severini and Shparlinski. Moreover, it is shown that in the class of unitary Cayley graphs there are only two of them allowing perfect state transfer.

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1 Introduction

In [5], the authors consider the arrangement problem of N interacting qubits in the network allowing the perfect state transfer of any quantum state between the qubits with the use of free evolution. It is known that the necessary condition for the perfect state transfer in qubit network is the periodicity of the system dynamics. Further research on this topic is performed by Saxena, Severini and Shparlinski ([9]) where it is proved that the dynamics of the quantum system is periodic if and only if the ratio of the difference of any two pairs of adjacency matrix eigenvalues is rational. It is also proved in [9] that the last condition implies that all circulant graph based networks have integer eigenvalues of the adjacency matrix. Circulant graphs are an important class of interconnection networks in parallel and distributed computing (see [3]). Therefore integral circulant graphs arises as potential candidates for modeling quantum spin networks that might enable the perfect state transfer between antipodal sites in a network.

Integral circulant graphs are first investigated by So [7], where it is given a characterization of this subclass of circulant graphs. Some other properties of integral circulant graphs, including the bound of the number of vertices, diameter and bipartiteness are later studied by Saxena, Severini and Shparlinski [9]. Furthermore, Stevanović, Petković and Bašić [4] improved the previous upper bound on the diameter and showed that the diameter of these graphs is at most $O(\ln \ln n)$.

Moreover, integral circulant graphs are the generalization of well-known class of unitary Cayley graphs. Various properties of unitary Cayley graphs were investigated in some recent papers. In the work of Berrizbeitia and Giudici [6], and in the later paper of Fuchs [1] the lower and upper bound on the size of the longest induced cycles are given. Klotz and Sander [8] determined the diameter, clique number, chromatic number and eigenvalues of unitary Cayley graphs. Bašić and Ilić [2] calculated the clique number for integral circulant graphs with exactly one and two divisors and also gave inequality for the general case.

The problem of existence of perfect state transfer in integral circulant graphs is already investigated by Saxena, Severini and Shparlinski [9]. They stated that there is no perfect state transfer in integral circulant graphs with an odd number of vertices. However the proof given in [9] does not hold in general case. The main drawback in this proof is the perfect state transfer existence condition which is not complete.

The aim of our paper is to provide simple and general characterization of the existence of perfect state transfer in integral circulant graphs, in terms of its eigenvalues. We give the correct proof of non existence of perfect state transfer in integral circulant graphs with an odd number of vertices. Moreover, we also prove that in the class of unitary Cayley graphs, there are only two graphs allowing perfect state transfer, one of them being already mentioned in [9].

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2 Integral circulant graphs

A *circulant graph* $G(n; S)$ is graph on vertices $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ such that each vertex i is adjacent to vertices $i + s$ for all $s \in S$. A set S is called a *symbol* of graph $G(n; S)$. Note that the degree of graph $G(n; S)$ is $\#S$. Graph is *integral* if all its eigenvalues are integers. Denote by

$$G_n(d) = \{k \ : \ \gcd(k, n) = d, \ 1 \leq k \leq n-1\},$$

and by D_n set of all positive divisors of n, less than n. Wasin So has characterised integral circulant graphs [7] by the following theorem

Theorem 1 (So 2006) *A circulant graph $G(n; S)$ is integral if and only if*

$$S = \bigcup_{d \in D} G_n(d),$$

for some set of divisors $D \subseteq D_n$.

Therefore an integral circulant graph $G(n; S)$ is defined by its order n and the set of divisors D. Such graphs are also known as *gcd-graphs* (see for example [8]). An integral circulant graph with n vertices defined with the set of divisors $D \subseteq D_n$ will be denoted by $\mathrm{ICG}_n(D)$. From Theorem 1 we have that the degree of an integral circulant graph is equal to $\deg \mathrm{ICG}_n(D) = \sum_{d \in D} \#G_n(d) = \sum_{d \in D} \varphi(n/d)$. Here $\varphi(n)$ denotes Euler-phi function [10]. The next theorem concerns the connectivity of an integral circulant graphs.

Theorem 2 (So 2006) *An integral circulant graph $\mathrm{ICG}_n(D)$, where $D = \{d_1, \ldots, d_k\}$ is connected if and only if $\gcd(n, d_1, \ldots, d_k) = 1$.*

The eigenvalues and eigenvectors of $\mathrm{ICG}_n(D)$ are given by [9]

$$\lambda_j = \sum_{s \in S} \omega_n^{js}, \quad v_j = [1 \ \omega_n^s \ \omega_n^{2s} \cdots \omega_n^{(n-1)s}],$$

where $\omega_n = e^{i\frac{2\pi}{n}}$ is the n-th root of unity.

Denote with $c(n, j)$ the following expression

$$c(j, n) = \mu(t_{n,j}) \frac{\varphi(n)}{\varphi(t_{n,j})}, \quad t_{n,j} = \frac{n}{\gcd(n, j)}, \tag{1}$$

where μ is Möbius function. Expression $c(j, n)$ is known as the *Ramanujan function* [10]. Eigenvalues λ_j can be expressed using the Ramanujan function as follows ([8], Theorem 16)

$$\lambda_j = \sum_{d \in D} c(j, n/d). \tag{2}$$

Let us observe the following properties of the Ramanujan function. These basic properties will be used in the rest of the paper.

Proposition 3 *For any positive integers n, j and d such that $d \mid n$, holds*

$$\begin{aligned}
c(0, n) &= \varphi(n), && (3)\\
c(1, n) &= \mu(n), && (4)\\
c(2, n) &= \begin{cases} \mu(n), & n \in 2\mathbb{N} + 1 \\ \mu(n/2), & n \in 4\mathbb{N} + 2 \\ 2\mu(n/2), & n \in 4\mathbb{N} \end{cases} && (5)\\
c(n/2, n/d) &= \begin{cases} \varphi(n/d), & d \in 2\mathbb{N} \\ -\varphi(n/d), & d \in 2\mathbb{N} + 1 \end{cases} && (6)
\end{aligned}$$

Proof. Directly by using relation (1). □

3 Perfect state transfer

For a given integral circulant graph $\mathrm{ICG}_n(D)$ we say that there is a *perfect state transfer* (PST) between vertices a and b if there is a positive real number t such that

$$|\langle a|e^{iAt}|b\rangle| = \left| \frac{1}{n} \sum_{l=0}^{n-1} e^{i\lambda_l t} \omega_n^{l(a-b)} \right| = 1. \tag{7}$$

From the triangle inequality we have obviously that $|\langle a|e^{iAt}|b\rangle| \leq 1$ holds, where equality is satisfied if and only if all summands in (7) have the same argument, i.e. are equal. In other words, there is a PST in $\mathrm{ICG}_n(D)$ if and only if

$$e^{i\lambda_0 t} = e^{i\lambda_1 t + i\frac{2\pi}{n}(a-b)} = \cdots = e^{i\lambda_{n-1}t + i\frac{2(n-1)\pi}{n}(a-b)}. \tag{8}$$

The last expression is equivalent to

$$\lambda_0 t \equiv_{2\pi} \lambda_1 t + \frac{2\pi}{n}(a-b) \equiv_{2\pi} \cdots \equiv_{2\pi} \lambda_{n-1} t + \frac{2(n-1)\pi}{n}(a-b).$$

Relation $\equiv_{2\pi}$ is defined by $A \equiv_{2\pi} B$ if $\frac{(A-B)}{2\pi} \in \mathbb{Z}$. Notice that (8) depends on a and b just as the function of $a-b$. Therefore we can, without loosing of generality, take $b=0$. By substracting adjacent congruences in previous equation and replacing $b=0$ we obtain that (8) is equivalent to the following $n-1$ conditions

$$(\lambda_{j+1} - \lambda_j)t_1 + \frac{a}{n} \in \mathbb{Z}, \quad j = 0, \ldots, n-2,$$

where $t_1 = t/(2\pi)$. From the last expression we can conclude that if there is a PST in $\mathrm{ICG}_n(D)$, then t is rational, i.e. there exists integers p and q such that $t_1 = p/q$ and $\gcd(p,q) = 1$.

The discussion above leads to the following result.

Theorem 4 *There exists PST in graph $\mathrm{ICG}_n(D)$ between a vertices a and 0 if and only if there are integers p and q such that $\gcd(p,q) = 1$ and*

$$\frac{p}{q}(\lambda_{j+1} - \lambda_j) + \frac{a}{n} \in \mathbb{Z}, \tag{9}$$

for all $j = 0, \ldots, n-2$.

Note that if there is PST in $\mathrm{ICG}_n(D)$, the following equation yields from (9)

$$\frac{p}{q}(\lambda_{j+2} - \lambda_j) + \frac{2a}{n} \in \mathbb{Z}, \quad j = 0, 1, \ldots, n-3 \tag{10}$$

The next corollary is derived from Theorem 4 and will be used as the criterion for the nonexistence of PST.

Corollary 5 *If $\lambda_j = \lambda_{j+1}$ for some $j = 0, \ldots, n-2$ then there is no PST in $\mathrm{ICG}_n(D)$ between any two vertices a and b.*

Proof. Without loosing of generality we can take $b = 0$. Theorem 4 yields that $a/n \in \mathbb{Z}$, i.e. $n \mid a$. This is impossible because $0 < a < n$. □

Theorem 6 *There is no PST in $\mathrm{ICG}_n(D)$ if n/d is odd for every $d \in D$. For n even, if there exists PST in $\mathrm{ICG}_n(D)$ between vertices a and 0 then $a = n/2$.*

Proof. First suppose that n/d is odd for every $d \in D$. Using Proposition 3 it is easy to prove that

$$\lambda_1 = \lambda_2 = \sum_{d \in D} \mu(n/d).$$

According to the Corollary 5 there is no PST in $\mathrm{ICG}_n(D)$.

Suppose now that n is even. Let us observe that $\gcd(n/2+1, n/d) = \gcd(n/2-1, n/d) \in \{1, 2\}$. Therefore holds $t_{n/d, n/2+1} = t_{n/d, n/2-1}$, i.e. $c(n/2-1, n/d) = c(n/2+1, n/d)$. Using the last expression we prove that

$$\lambda_{n/2-1} = \sum_{d \in D} c(n/2-1, n/d) = \sum_{d \in D} c(n/2+1, n/d) = \lambda_{n/2+1}$$

Again using (10) we have that $(2a)/n \in \mathbb{Z}$, which is possible just for $a = n/2$. □

According to the last theorem, PST is possible in $\mathrm{ICG}_n(D)$ just for n even and between vertices $a = n/2$ and 0 (i.e. between b and $n/2 + b$ as mentioned in [9]). Therefore, in the rest of the paper we will suppose that n is even and $a = n/2$. Relation (9) now becomes

$$\frac{p(\lambda_{j+1} - \lambda_j)}{q} + \frac{1}{2} \in \mathbb{Z}. \tag{11}$$

For a given prime number p and integer n, denote by $S_p(n)$ the maximal number α such that $p^\alpha \mid n$.

Lemma 7 *There exists PST in* $\mathrm{ICG}_n(D)$*, if and only if there exists number* $m \in \mathbb{N}_0$ *such that the following holds for all* $j = 0, 1, \ldots, n-2$

$$S_2(\lambda_{j+1} - \lambda_j) = m. \tag{12}$$

Proof. Let $\lambda_{j+1} - \lambda_j = 2^{s_j} m_j$ where $s_j = S_2(\lambda_{j+1} - \lambda_j) \geq 0$ and m_j is odd integer for each $j = 0, 1, \ldots, n-2$.
($\Rightarrow$:) Suppose that $\mathrm{ICG}_n(D)$ has PST. According to the Theorem 4, there exist relatively prime integers p, q such that (11) holds. Rewrite relation (8) in the following form

$$\frac{2^{s_j+1} p m_j + q}{2q} \in \mathbb{Z}. \tag{13}$$

From the last expression we can conclude that $q \mid 2^{s_j+1} m_j$ (because $\gcd(p, q) = 1$) and $2 \mid q$. Furthermore there must exist non-negative integers s_q and $m_q \in 2\mathbb{N} + 1$ such that $q = 2^{s_q+1} m_q$ where $s_q \leq s_j$ and $m_q \mid m_j$ for each $j = 0, 1, \ldots, n-2$. By exchanging in (13) we obtain

$$\frac{2^{s_j - s_q} p \frac{m_j}{m_q} + 1}{2} \in \mathbb{Z},$$

which directly implies that $s_j = s_q = S_2(q) - 1$. Now by putting $m = S_2(q) - 1$ we obtain (12).
($\Leftarrow$:) Now suppose that (12) is valid. Put $q = 2^{m+1}$ and $p = 1$. Then it holds

$$\frac{p(\lambda_{j+1} - \lambda_j)}{q} + \frac{1}{2} = \frac{m_j + 1}{2} \in \mathbb{Z},$$

for every $j = 0, 1, \ldots, n-2$. According to Theorem 4 there is PST in $\mathrm{ICG}_n(D)$. □

4 PST in unitary Cayley graphs

The unitary Cayley graphs are the special case of integral circulant graphs with $D = \{1\}$. Note that if $D = \{d\}$ then $\mathrm{ICG}_n(D)$ is connected if and only if $d = 1$ (according to the Theorem 2). Hence the only integral circulant graphs with $D = \{d\}$ that might have PST are the unitary Cayley graphs. The main result of this section is the nonexistence of the perfect state transfer in unitary Cayley graphs, except in K_2 and C_4.

The next proposition resolves the case when both n and $n/2$ are not square-free.

Proposition 8 *If both n and $n/2$ are not square-free integers, there is no PST in* $\mathrm{ICG}_n(\{1\})$.

Proof. According to the Proposition 3 we have that $\lambda_1 = \lambda_2 = 0$ because $\mu(n) = \mu(n/2) = 0$. Applying the Corollary 5 yields that there is no PST in $\mathrm{ICG}_n(\{1\})$. □

Next consider the case when $n = 2s$ where s is square-free integer.

Lemma 9 *If $n = 2s$ where s is square-free integer, the only unitary Cayley graphs that have PST are K_2 and C_4.*

Proof. Suppose that $\mathrm{ICG}_n(\{1\})$ has PST. It holds $\lambda_1 = \mu(n)$ and

$$\lambda_2 = \begin{cases} \mu(s), & s \in 2\mathbb{N} + 1 \\ 2\mu(s), & s \in 2\mathbb{N} \end{cases}.$$

Moreover since $\mu(s) = -\mu(n)$ for $s \in 2\mathbb{N} + 1$ and $\mu(n) = 0$ for $s \in 2\mathbb{N}$ we can conclude that $|\lambda_1 - \lambda_2| = 2$. From relation (9) and using that $a = n/2$ we obtain that

$$\frac{2p}{q} + \frac{1}{2} \in \mathbb{Z} \quad \text{or} \quad -\frac{2p}{q} + \frac{1}{2} \in \mathbb{Z}. \tag{14}$$

Suppose that the first statement is valid. Then $\frac{4p+q}{2q} \in \mathbb{Z}$ and therefore $q \mid 4p$ which implies that $q \mid 4$. On the other hand we have $2 \mid q$. There are only two possibilities for q, $q = 2$ and $q = 4$. Case $q = 2$ is impossible because $(2p+1)/2 \notin \mathbb{Z}$ for every $p \in \mathbb{Z}$. There must hold $q = 4$. The same conclusion can be derived analogously by assuming that the second statement in (14) is valid.

Let $n \geq 3$ and $s \in 2\mathbb{N}+1$. Note that $\lambda_s = -\varphi(n)$ and $\lambda_{s-1} = -\mu(n)$. Whereas $\varphi(n)$ is even and $\mu(n)$ is odd ($n = 2s$ is square free), it holds that $|\lambda_{s-1}-\lambda_s|$ is odd. This is contradiction with Lemma 7 since $S_2(|\lambda_1-\lambda_2|) \neq S_2(|\lambda_{s-1}-\lambda_s|)$.

Let $n > 4$ and $s \in 2\mathbb{N}$. Then $\lambda_s = -\varphi(n) \in 4\mathbb{N}$ and $\lambda_{s-1} = 0$. This means that $S_2(|\lambda_{s-1} - \lambda_s|) \geq 2$ and since $S_2(|\lambda_1 - \lambda_2|) = 1$ this is again contradiction with Lemma 7.

Conditions $n \leq 2$ and $s \in 2\mathbb{N} + 1$ are satisfied only for $n = 2$ where by direct computation of eigenvalues and verification in (9) we obtain that $\mathrm{ICG}_2(\{1\})$ has PST. Note that $\mathrm{ICG}_2(\{1\}) = K_2$.

Finally, conditions $n \leq 4$ and $s \in 2\mathbb{N}$ are satisfied just for $n = 4$. Also by direct verification we obtain that $\mathrm{ICG}_4(\{1\})$ has PST. In such case holds $\mathrm{ICG}_4(\{1\}) = C_4$. $\square$

According to Theorem 6, Proposition 8 and Lemma 9 the following main result of this section directly holds.

Theorem 10 *The only unitary Cayley graphs that have PST are K_2 and C_4.*

5 Conclusion

We developed the simple and general characterization of integral circulant graphs allowing perfect state transfer in terms of their spectra. Its existence (other than trivial examples) was asked by Saxena, Severini and Shparlinski in [9]. It turns out that there is no such graphs in the class of unitary Cayley graphs, except for K_2 and C_4.

We also implemented the exhaustive computer search algorithm for finding integral circulant graphs other than unitary Cayley graphs which have a perfect state transfer. Such graphs with the least number of vertices are $\mathrm{ICG}_8(\{1, 2\})$ and $\mathrm{ICG}_8(\{1, 4\})$.

These examples provide a positive answer to the question posed in [9]. Moreover, they suggest that two classes of integral circulant graphs $\mathrm{ICG}_n(\{1, n/4\})$ and $\mathrm{ICG}_n(\{1, n/2\})$ allow PST where $n \in 8\mathbb{N}$. An answer to the last question might be the future research on this topic. Also it would be interesting to see whether there are more graphs $\mathrm{ICG}_n(D)$ with $|D| = 2$ which have the perfect state transfer.

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