The Hankel Transform of the Sum of Consecutive Generalized Catalan Numbers

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Abstract. We discuss the properties of the Hankel transformation of a sequence whose elements are the sums of consecutive generalized Catalan numbers.

1. Introduction

Let \( A = \{a_0, a_1, a_2, \ldots\} \) be a sequence of real numbers. The Hankel matrix generated by \( A \) is the infinite matrix \( H = [h_{i,j}] \), where \( h_{i,j} = a_{i+j-2} \), i.e.,

\[
H = \begin{bmatrix}
a_0 & a_1 & a_2 & a_3 & \ldots \\
a_1 & a_2 & a_3 & a_4 & \ldots \\
a_2 & a_3 & a_4 & a_5 & \ldots \\
a_3 & a_4 & a_5 & a_6 & \ldots \\
a_4 & a_5 & a_6 & a_7 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}
\] (1)

The Hankel matrix \( H_n \) of order \( n \) is the upper-left \( n \times n \) submatrix of \( H \) and the Hankel determinant of order \( n \) of \( A \), denoted by \( h_n \), is the determinant of the corresponding Hankel matrix.

For a given sequence \( A = \{a_0, a_1, a_2, \ldots\} \), the Hankel transform of \( A \) is the corresponding sequence of Hankel determinants \( \{h_0, h_1, h_2, \ldots\} \) (see Layman [?]).

The elements of the sequence in which we are interested are the sums of two adjacent generalized Catalan numbers with parameter \( L \):

\[
a_n = a_n(L) = c(n; L) + c(n + 1; L) \quad (n = 0, 1, 2, \ldots),
\] (2)

\[
c(n; L) = T(2n, n; L) - T(2n, n - 1; L),
\] (3)
with

\[ T(n, k; L) = \sum_{j=0}^{n-k} \binom{k}{j} \binom{n-k}{j} L^j. \quad (4) \]

Here

\[ a_0 = L + 1. \quad (5) \]

**Example 1.1.** Let \( L = 1 \). Vandermonde’s convolution identity implies that

\[ \binom{n}{k} = \sum_j \binom{k}{j} \binom{n-k}{j}. \]

Hence we have

\[ T(2n, n; 1) = \binom{2n}{n}, \quad T(2n, n-1; 1) = \binom{2n}{n-1}, \]

wherefrom we get Catalan numbers

\[ c(n) = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n} \]

and

\[ a_n = c(n) + c(n+1) = \frac{(2n)!(5n+4)}{n!(n+2)!} \quad (n = 0, 1, 2, \ldots). \]

In the paper [?] the authors have proved that the Hankel transform of \( a_n \) equals sequence of Fibonacci numbers with odd indices

\[ h_n = F_{2n+1} = \frac{1}{\sqrt{5}} \frac{1}{2^{n+1}} \left\{ (\sqrt{5} + 1)(3 + \sqrt{5})^n + (\sqrt{5} - 1)(3 - \sqrt{5})^n \right\}. \]

**Example 1.2.** For \( L = 2 \) we get like \( a_n(2) \) the next numbers

\[ 3, 8, 28, 112, 484, \ldots, \]

and the Hankel transform \( h_n \):

\[ 3, 20, 272, 7424, 405504, \ldots. \]

One of us, P. Barry conjectured that

\[ h_n(2) = 2^{\frac{n^2-n-2}{2}} \left\{ (2 + \sqrt{2})^{n+1} + (2 - \sqrt{2})^{n+1} \right\}. \]

In general, he made the conjecture, which we will prove through this paper.

**Theorem 1.1.** (The main result) *For the generalized Pascal triangle associated to the sequence \( n \mapsto L^n \), the Hankel transform of the sequence

\[ c(n; L) + c(n + 1; L) \]
is given by
\[
\begin{align*}
    h_n &= \frac{L^{(n^2-n)/2}}{2^{n+1}\sqrt{L^2 + 4}} \\
    &\cdot \left\{ \left( \sqrt{L^2 + 4} + L \right) \left( \sqrt{L^2 + 4} + L + 2 \right)^n + \left( \sqrt{L^2 + 4} - L \right) \left( L + 2 - \sqrt{L^2 + 4} \right)^n \right\}.
\end{align*}
\] (6)

From now till the end, let us denote by
\[
    \xi = \sqrt{L^2 + 4}, \quad t_1 = L + 2 + \xi, \quad t_2 = L + 2 - \xi.
\] (7)

Now, we can write
\[
    h_n = \frac{L^{n(n-1)/2}}{2^{n+1}\xi} \cdot \left( (\xi + L)t_1^n + (\xi - L)t_2^n \right). \quad (8)
\]

Or, introducing
\[
    \varphi_n = t_1^n + t_2^n, \quad \psi_n = t_1^n - t_2^n \quad (n \in \mathbb{N}_0),
\] (9)

the final statement can be expressed by
\[
    h_n = \frac{L^{n(n-1)/2}}{2^{n+1}\xi} \cdot \left( L\psi_n + \xi\varphi_n \right). \quad (10)
\]

Lemma 1.1. The values \( \varphi_n \) and \( \psi_n \) satisfy the relations
\[
\begin{align*}
    \varphi_j \cdot \varphi_k &= \varphi_{j+k} + (4L)^j \varphi_{k-j}, \quad \psi_j \cdot \psi_k = \varphi_{j+k} - (4L)^j \varphi_{k-j} \quad (0 \leq j \leq k) \quad (11)
\end{align*}
\]

and
\[
\begin{align*}
    \varphi_j \cdot \psi_k &= \varphi_{j+k} + (4L)^j \psi_{k-j}, \quad \psi_j \cdot \varphi_k = \psi_{j+k} - (4L)^j \psi_{k-j} \quad (0 \leq j \leq k). \quad (12)
\end{align*}
\]

Corollary 1.1. The function \( h_n = h_n(L) \) is the next polynomial
\[
    h_n(L) = 2^{-n} L^{n(n-1)/2} \cdot \left\{ \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2i+1} L(L + 2)^{n-2i-1}(L^2 + 4)^i + \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} (L + 2)^{n-2i}(L^2 + 4)^i \right\}.
\]
Proof. By previous notation, we can write
\[(L + \xi)(L + 2 + \xi)^n - (L - \xi)(L + 2 - \xi)^n\]
\[= (L + \xi) \sum_{k=0}^{n} \binom{n}{k} (L + 2)^{n-k} \xi^k - (L - \xi) \sum_{k=0}^{n} (-1)^k \binom{n}{k} (L + 2)^{n-k} \xi^k\]
\[= \sum_{k=0}^{n} (1 - (-1)^k) \binom{n}{k} L(L + 2)^{n-k} \xi^k + \sum_{k=0}^{n} (1 + (-1)^k) \binom{n}{k} (L + 2)^{n-k} \xi^{k+1}\]
\[= 2 \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2i+1} L(L + 2)^{n-2i-1} \xi^{2i+1} + 2 \sum_{i=0}^{\left\lceil \frac{n}{2} \right\rceil} \binom{n}{2i} L(L + 2)^{n-2i} \xi^{2i+1}\]
\[= 2\xi \left\{ \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2i+1} L(L + 2)^{n-2i-1} \xi^{2i} + \sum_{i=0}^{\left\lceil \frac{n}{2} \right\rceil} \binom{n}{2i} L(L + 2)^{n-2i} \xi^{2i} \right\},\]
wherefrom immediately follows the expression for \(h_n\).

2. THE GENERATING FUNCTION

The Jacobi polynomials are given by
\[P_n^{(a,b)}(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n + a}{k} \binom{n + b}{n - k} (x - 1)^{n-k} (x + 1)^k \quad (a, b > -1).\]

Also, they can be written in the form
\[P_n^{(a,b)}(x) = \left(\frac{x - 1}{2}\right)^n \sum_{k=0}^{n} \binom{n + a}{k} \binom{n + b}{n - k} (x + 1)^k.\]

From the fact
\[L = \frac{x + 1}{x - 1} \iff x = \frac{L + 1}{L - 1} \quad (x \neq 1, \ L \neq 1),\]
we conclude that
\[T(2n, n; L) = (L - 1)^n \cdot P_n^{(0,0)}\left(\frac{L+1}{L-1}\right)\]
and
\[T(2n + 2, n; L) = (L - 1)^n \cdot P_n^{(2,0)}(\frac{L+1}{L-1}).\]

The generating function \(G(x, t)\) for the Jacobi polynomials is
\[G^{(a,b)}(x, t) = \sum_{n=0}^{\infty} P_n^{(a,b)}(x) t^n = \frac{2^{a+b}}{\phi(1-t+\phi)^a \cdot (1+t+\phi)^b}, \quad (13)\]
where
\[\phi = \phi(x, t) = \sqrt{1 - 2xt + t^2}.\]
Now,
\[
\sum_{n=0}^{\infty} T(2n, n; L) t^n = \sum_{n=0}^{\infty} P_n^{(0,0)} \left( \frac{t+1}{L-1} \right) ((L-1)t)^n = G^{(0,0)} \left( \frac{t+1}{L-1}, (L-1)t \right),
\]
\[
\sum_{n=0}^{\infty} T(2n + 2, n; L) t^n = \sum_{n=0}^{\infty} P_n^{(2,0)} \left( \frac{t+1}{L-1} \right) ((L-1)t)^n = G^{(2,0)} \left( \frac{t+1}{L-1}, (L-1)t \right).
\]
Also,
\[
\sum_{n=0}^{\infty} T(2n, n-1; L) t^n = t \cdot \left\{ G^{(2,0)} \left( \frac{t+1}{L-1}, (L-1)t \right) - 1 \right\},
\]
\[
\sum_{n=0}^{\infty} T(2n + 2, n + 1; L) t^n = \frac{1}{t} \cdot \left\{ G^{(0,0)} \left( \frac{t+1}{L-1}, (L-1)t \right) - 1 \right\}.
\]
The generating function \( G(t; L) \) for the sequence \( \{a_n\}_{n \geq 0} \) is given by
\[
G(t; L) = \sum_{n=0}^{\infty} a_n t^n = \frac{t+1}{t} G^{(0,0)} \left( \frac{t+1}{L-1}, (L-1)t \right) - (t + 1)G^{(2,0)} \left( \frac{t+1}{L-1}, (L-1)t \right) - \frac{1}{t}. \tag{14}
\]
The function
\[
\rho(t; L) = \phi \left( \frac{t+1}{L-1}, (L-1)t \right) = \sqrt{1 - 2(L+1)t + (L-1)^2t^2} \tag{15}
\]
has domain
\[
D_\rho = \left( -\infty, \frac{1 - 2\sqrt{L} + L}{1 - 2L + L^2} \right) \cup \left( \frac{1 + 2\sqrt{L} + L}{1 - 2L + L^2}, +\infty \right) \quad (L \neq 1),
\]
or
\[
D_\rho = \left( -\infty, 1/4 \right) \quad (L = 1).
\]
**Theorem 2.1.** The generating function \( G(t; L) \) for the sequence \( \{a_n\}_{n \geq 0} \) is
\[
G(t; L) = \frac{t+1}{\rho(t; L)} \left\{ \frac{1}{t} - \frac{4}{(1 - (L-1)t + \rho(t; L))^2} \right\} - \frac{1}{t}. \tag{16}
\]
**Example 2.1.** For \( L = 1 \), we get
\[
G(t; 1) = \sum_{n=0}^{\infty} a_n(1) t^n = \frac{1}{t} \left( \frac{(1 - \sqrt{1-4t})(1+t)}{2t} - 1 \right). \tag{17}
\]
and for \( L = 2 \), we find
\[
G(t; 2) = \sum_{n=0}^{\infty} a_n(2) t^n = \frac{-1}{t} + \frac{t+1}{\sqrt{t^2 - 6t + 1}} \left\{ \frac{1}{t} - \frac{4}{(1 - t + \sqrt{t^2 - 6t + 1})^2} \right\}. \tag{18}
\]
It is known (for example, see Krattenthaler [??]) that the Hankel determinant $h_n$ of order $n$ of the sequence $\{a_n\}_{n \geq 0}$ equals

$$h_n = a_0^n \beta_1^{n-1} \beta_2^{n-2} \cdots \beta_{n-2}^{2} \beta_{n-1}, \quad (19)$$

where $\{\beta_n\}_{n \geq 1}$ is the sequence given by:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{a_0}{1 + \alpha_0 x - \frac{\beta_1 x^2}{1 + \alpha_1 x - \frac{\beta_2 x^2}{1 + \alpha_2 x - \cdots}}} \quad (20)$$

The sequences $\{\alpha_n\}_{n \geq 0}$ and $\{\beta_n\}_{n \geq 1}$ are the coefficients in the recurrence relation

$$Q_{n+1}(x) = (x - \alpha_n)Q_n(x) - \beta_n Q_{n-1}(x), \quad (21)$$

where $\{Q_n(x)\}_{n \geq 0}$ is the monic polynomial sequence orthogonal with respect to the functional $L$ determined by

$$L[x^n] = a_n \quad (n = 0, 1, 2, \ldots). \quad (22)$$

In the next section this functional will be constructed for the sum of consecutive generalized Catalan numbers.

3. THE WEIGHT FUNCTION CORRESPONDING TO THE FUNCTIONAL

We would like to express $L[f]$ in the form:

$$L[f(x)] = \int_{R} f(x) d\psi(x),$$

where $\psi(x)$ is a distribution, or, even more, to find the weight function $w(x)$ such that $w(x) = \psi'(x)$.

Denote by $F(z; L)$ the function

$$F(z; L) = \sum_{k=0}^{\infty} a_k z^{-k-1}.$$  

From the generating function (??), we have:

$$F(z; L) = z^{-1} G(z^{-1}; L). \quad (23)$$

Example 3.1. From (??), we have:

$$F(z; 1) = z^{-1} G(z^{-1}; 1) = \frac{1}{2} \left\{ z - 1 - (z + 1) \sqrt{1 - \frac{4}{z}} \right\}.$$
Example 3.2. From (??), we have:

\[ F(z; 2) = \frac{-1}{2z} \left\{ 1 + z \left( 2 - z + (z + 1) \sqrt{1 - \frac{6}{z} + \frac{1}{z^2}} \right) \right\}. \]

\[ \int F(z; 2) dz = z + \frac{1}{4} z(z - 1)\rho(1/z, 2) + \log(z) \]
\[ - \frac{1}{2} \log \left( 1 + z(\rho(1/z, 2) - 3) \right) - \frac{7}{2} \log(z - 3 + z\rho(1/z, 2)). \]

When we replace relation (8) in previous relation and after some simplifications we obtain that

\[ F(z; L) = -1 + \frac{2(z + 1)}{L - 1 + z + \sqrt{L^2 + (z - 1)^2 - 2L(z + 1)}} \]
\[ = -1 + \frac{2(z + 1)}{L - 1 + z(1 + z\rho(1/z, L))} \]

Denote \( R(z; L) = z\rho(1/z, L) = \sqrt{L^2 + (z - 1)^2 - 2L(z + 1)} \). From the theory of distribution functions (see Chihara [?]), we have Stieltjes inversion formula

\[ \psi(t) - \psi(0) = -\frac{1}{\pi} \lim_{y \to 0^+} \int_0^t \text{Re} F(x + iy; L) dx. \quad (24) \]

using which we can calculate the distribution function \( \psi(x) \) and weight function \( w(x) = \psi'(x) \). It can be shown that holds:

\[ \int F(z; L) dz = \frac{1}{4} \left[ z^2 - 2Lz - (z - L + 1)R(z; L) - l_1(z) + l_2(z) \right] \quad (25) \]

where we denoted:

\[ l_1(z) = 2(3L + 1) \log \left[ z - (L + 1) + R(z; L) \right] \]
\[ l_2(z) = 2(L - 1) \log \left[ \frac{-(L - 1)R(z; L) - (L - 1)^2 + z(L + 1)}{z^2(L - 1)^3} \right] \]

Denote with \( \mathcal{F}(z; L) = \int F(z; L) dz \). Now we will replace \( z = x + iy \) and find a limit \( \lim_{y \to 0^+} \mathcal{F}(x + iy; L) \). First rewrite function \( R(z; L) \) in the following form:

\[ R(z; L) = \sqrt{(z - L - 1)^2 - 4L} \]

Now replace \( z = x + iy \) and let \( y \) tends to 0\. Then we have:

\[ R(x; L) = \lim_{y \to 0^+} R(x + iy; L) = \begin{cases} \frac{i\sqrt{4L - (x - L - 1)^2}}{(x - L - 1)^2 - 4L}, & x \in (a, b); \\ \frac{\sqrt{(x - L - 1)^2 - 4L}}{\sqrt{(x - L - 1)^2 - 4L}}, & otherwise, \end{cases} \]

where

\[ a = (\sqrt{L} - 1)^2, \quad b = (\sqrt{L} + 1)^2. \quad (26) \]
In the case when \( x \notin \left( \sqrt{L} - 1, \sqrt{L} + 1 \right) \), value \( R(x; L) \) is real. Therefore we can calculate imaginary part of \( \mathcal{F}(x; L) = \lim_{y \to 0^+} \mathcal{F}(x + iy; L) \):

\[
\Re \mathcal{F}(x; L) = \Re [l_2(x) - l_1(x)] = 0.
\]

Otherwise, if \( x \in \left( \sqrt{L} - 1, \sqrt{L} + 1 \right) \) we have that:

\[
l_1(x) = 2(3L + 1) \log \left[ x - (L + 1) \pm i \sqrt{4L - (x - L - 1)^2} \right]
\]

\[
\Im l_1(x) = \begin{cases} 
2(3L + 1) \arctan \frac{\sqrt{4L - (x - L - 1)^2}}{x - (L + 1)}, & x \geq L + 1; \\
2(3L + 1) \left( \pi + \arctan \frac{\sqrt{4L - (x - L - 1)^2}}{x - (L + 1)} \right), & x < L + 1
\end{cases}
\]

\[
l_2(x) = 2(L - 1) \log \left[ \frac{-(L-1)^2 + 2x(L+1) - i(L-1) \sqrt{4L - (x - L - 1)^2}}{x^2(L-1)^3} \right]
\]

\[
\Im l_1(x) = \begin{cases} 
2(L - 1) \left( 2\pi + \arctan \frac{x(L+1) - (L-1)^2}{\sqrt{4L - (x - L - 1)^2}} \right), & x \geq \frac{(L-1)^2}{L+1}; \\
2(L - 1) \left( \pi + \arctan \frac{x(L+1) - (L-1)^2}{\sqrt{4L - (x - L - 1)^2}} \right), & x < \frac{(L-1)^2}{L+1}
\end{cases}
\]

After substituting all considered cases in (7) we finally obtain value

\[
\Re \mathcal{F}(x; L) = \lim_{y \to 0^+} \Re \mathcal{F}(x + iy; L) = \Re l_2(x) - \Re l_1(x) - (x - L + 1) \sqrt{4L - (x - L - 1)^2}
\]

From the relation (16) we conclude that

\[
\omega(x; L) = \psi'(x; L) = -\frac{1}{\pi} \frac{d}{dx} \Re \mathcal{F}(x; L) \quad (27)
\]

and finally we obtain \( \omega(x; L) \):

\[
\omega(x; L) = \frac{1}{2\pi} (1 + \frac{1}{x}) \sqrt{4L - (x - L - 1)^2} = \frac{\sqrt{L}}{\pi} \left( 1 + \frac{1}{x} \right) \sqrt{1 - \left( \frac{x - L - 1}{2\sqrt{L}} \right)^2} \quad (28)
\]

Previous formula holds for \( x \in (a, b) \), and otherwise we have \( \omega(x; L) = 0 \).

4. Determining the three-term recurrence relation

The crucial moment in our proof of the conjecture is to determine the sequence of polynomials \( \{Q_n(x)\} \) orthogonal with respect to the weight \( w(x; L) \) given by \( ??? \) on the interval \( (a, b) \) and to find the sequences \( \{\alpha_n(x)\} \{\beta_n(x)\} \) in the three-term recurrence relation.
Example 4.1. For \( L = 4 \) we get
\[
Q_0(x) = 1,
\]
\[
Q_1(x) = x - \frac{24}{5},
\]
\[
Q_2(x) = x^2 - \frac{127}{13} x + \frac{256}{13},
\]
\[
Q_3(x) = x^3 - \frac{541}{17} x^2 + \frac{1096}{17} x - \frac{1344}{17},
\]
wherefrom\[ \alpha_0 = \frac{24}{5}, \quad \beta_0 = 5, \quad \alpha_1 = \frac{323}{65}, \quad \beta_1 = \frac{104}{25}, \quad \alpha_2 = \frac{1104}{221}, \quad \beta_2 = \frac{680}{169}. \]
Hence\[ h_1 = a_0 = 5, \quad h_2 = a_0^2 \beta_1 = 104, \quad h_3 = a_0^3 \beta_1^2 \beta_2 = 5^3 \left( \frac{104}{25} \right)^2 \frac{680}{169} = 8704. \]

At the beginning, we will notice that in the definition of the weight appears the square root member.

That's why, let us consider the monic orthogonal polynomials \( \{S_n(x)\} \) with respect to the \( p^{(1/2,1/2)}(x) = \sqrt{1-x^2} \) on the interval \((-1,1)\). These polynomials are monic Chebyshev polynomials of the second kind:
\[
S_n(x) = \frac{\sin((n+1) \arccos x)}{2^n \cdot \sqrt{1-x^2}}
\]
They satisfy the three-term recurrence relation (Chihara [?]):
\[
S_{n+1}(x) = (x - \alpha_n^*) S_n(x) - \beta_n^* S_{n-1}(x) \quad (n = 0, 1, \ldots),
\]
with initial values
\[
S_{-1}(x) = 0, \quad S_0(x) = 1,
\]
where
\[
\alpha_n^* = 0 \quad (n \geq 0) \quad \text{and} \quad \beta_0^* = \frac{\pi}{2}, \quad \beta_n^* = \frac{1}{4} \quad (n \geq 1).
\]
If we use the weight function \( \hat{w}(x) = (x - c) p^{(1/2,1/2)}(x) \), then the corresponding coefficients \( \hat{\alpha}_n \) and \( \hat{\beta}_n \) can be evaluated as follows (see, for example, Gautschi [?])
\[
\lambda_n = S_n(c),
\]
\[
\hat{\alpha}_n = c - \frac{\lambda_{n+1}}{\lambda_n} - \beta_{n+1}^* \frac{\lambda_n}{\lambda_{n+1}},
\]
\[
\hat{\beta}_n = \beta_n^* \frac{\lambda_{n-1} \lambda_{n+1}}{\lambda_n^2} \quad (n \in \mathbb{N}_0).
\]
From the relation (21), we conclude that the sequence \( \{\lambda_n\}_{n \in \mathbb{N}} \) satisfies the following recurrence relation:
\[
4\lambda_{n+1} - 4c \lambda_n + \lambda_{n-1} = 0 \quad (\lambda_{-1} = 0; \quad \lambda_0 = 1).
\]
The characteristic equation
\[ 4z^2 - 4cz + 1 = 0 \]
has the solutions
\[ z_{1,2} = \frac{1}{2} \left( c \pm \sqrt{c^2 - 1} \right). \]
and the integral solution of (23) is
\[ \lambda_n = E_1 z_1^n + E_2 z_2^n \quad (n \in \mathbb{N}). \]
We evaluate values \(E_1\) and \(E_2\) from the initial conditions \((\lambda_{-1} = 0; \ \lambda_0 = 1)\).

In order to solve our problem, we will choose \(c = -\frac{L+2}{2\sqrt{L}}\). Hence
\[ z_k = \frac{-t_k}{4\sqrt{L}} \quad (k = 1, 2), \quad \text{where} \quad t_{1,2} = L + 2 \pm \sqrt{L^2 + 4}. \]
Finally, we obtain:
\[ \lambda_n = \frac{(-1)^n}{2 \cdot 4^n L^{3/2} \sqrt{L^2 + 4} } \left( t_1^{n+1} - t_2^{n+1} \right) \quad (\lambda = -1, 0, 1, \ldots), \]
i.e.,
\[ \lambda_n = \frac{(-1)^n}{2 \cdot 4^n L^{3/2} \xi} \psi_{n+1} \quad (\lambda = -1, 0, 1, \ldots). \]
After replacing in (22), we obtain:
\[ \hat{\alpha}_n = \frac{-L + 2}{2\sqrt{L}} + \frac{1}{4\sqrt{L}} \cdot \frac{\psi_{n+2}}{\psi_{n+1}} + \sqrt{L} \cdot \psi_{n+1}, \quad \hat{\beta}_n = \frac{\psi_n \psi_{n+2}}{4\psi_{n+1}^2}. \]
If a new weight function \(\tilde{w}(x)\) is introduced by
\[ \tilde{w}(x) = \tilde{w}(ax + b) \]
then we have
\[ \tilde{\alpha}_n = \frac{\hat{\alpha}_n - b}{a}, \quad \tilde{\beta}_n = \frac{\hat{\beta}_n}{a^2} \quad (n \geq 0). \]
Now, by using \(x \mapsto \frac{x-L-1}{2\sqrt{L}}\), i.e., \(a = \frac{1}{2\sqrt{L}}\) and \(b = -\frac{L+1}{2\sqrt{L}}\), we have the weight function
\[ \tilde{w}(x) = \tilde{w}(\frac{x-L-1}{2\sqrt{L}}) = \frac{1}{2} \left( \frac{x-L-1}{2\sqrt{L}} + \frac{L+2}{2\sqrt{L}} \right) \sqrt{1 - \left( \frac{x-L-1}{2\sqrt{L}} \right)^2}. \]
Thus
\[ \tilde{\alpha}_n = -1 + \frac{1}{2} \cdot \frac{\psi_{n+2}}{\psi_{n+1}} + 2L \cdot \frac{\psi_{n+1}}{\psi_{n+2}} \quad (n \in \mathbb{N}_0), \quad (34) \]
and
\[ \tilde{\beta}_0 = (L + 2) \frac{\pi}{2}, \quad \tilde{\beta}_n = \frac{L \psi_n \psi_{n+2}}{\psi_{n+1}^2} \quad (n \in \mathbb{N}). \quad (35) \]

Example 4.2. For \( L = 4 \) we get

\[
\begin{align*}
P_0(x) &= 1, \quad \|P_0\|^2 = 3\pi, \\
P_1(x) &= x - \frac{17}{3}, \quad \|P_1\|^2 = \frac{32\pi}{3}, \\
P_2(x) &= x^2 - \frac{43}{4}x + \frac{101}{4}, \quad \|P_2\|^2 = 42\pi, \\
P_3(x) &= x^3 - \frac{331}{21}x^2 + \frac{1579}{21}x - \frac{2189}{21}, \quad \|P_3\|^2 = \frac{3520\pi}{21},
\end{align*}
\]

wherefrom

\[
\tilde{\alpha}_0 = \frac{17}{3}, \quad \tilde{\beta}_0 = 3\pi, \quad \tilde{\alpha}_1 = \frac{61}{12}, \quad \tilde{\beta}_1 = \frac{32}{9}, \quad \tilde{\alpha}_2 = \frac{421}{84}, \quad \tilde{\beta}_2 = \frac{63}{16}.
\]

Introducing the weight

\[ \bar{w}(x) = \frac{2L}{\pi} \tilde{w}(x) \]

will not change the monic polynomials and their recurrence relations, only it will multiply the norms by the factor \( 2L/\pi \), i.e.

\[
\|\tilde{P}_k\|_{\bar{w}}^2 = \int_a^b \tilde{P}_k(x) \bar{w}(x) \, dx = \frac{2L}{\pi} \|P_k\|^2 \quad (k \in \mathbb{N}_0),
\]

\[ \tilde{\beta}_0 = L(L + 2), \quad \tilde{\beta}_k = \tilde{\beta}_k \quad (k \in \mathbb{N}), \quad \tilde{\alpha}_k = \tilde{\alpha}_k \quad (k \in \mathbb{N}_0). \]

Here is

\[ \tilde{\beta}_0 \tilde{\beta}_1 \cdots \tilde{\beta}_{n-1} = \frac{L^n}{2} \cdot \frac{\psi_{n+1}}{\psi_n}. \quad (36) \]

In the book [?], W. Gautschi has treated the next problem: If we know all about the MOPS orthogonal with respect to \( \tilde{w}(x) \) what can we say about the sequence \( \{Q_n(x)\} \) orthogonal with respect to a weight

\[ w_d(x) = \frac{\bar{w}(x)}{x - d} \quad (d \notin \text{support}(\tilde{w})). \]

W. Gautshi has proved that, by the auxiliary sequence

\[ r_{-1} = - \int_{\mathbb{R}} w_d(x) \, dx, \quad r_n = d - \tilde{\alpha}_n - \frac{\tilde{\beta}_n}{r_{n-1}} \quad (n = 0, 1, \ldots), \]

it can be determined

\[
\begin{align*}
\alpha_{d,0} &= \tilde{\alpha}_0 + r_0, \quad &\alpha_{d,k} &= \tilde{\alpha}_k + r_k - r_{k-1}, \\
\beta_{d,0} &= -r_{-1}, \quad &\beta_{d,k} &= \tilde{\beta}_{k-1} \frac{r_{k-1}}{r_{k-2}} \quad (k \in \mathbb{N}).
\end{align*}
\]
In our case it is enough to take $d = 0$ to get the final weight
\[ w(x) = \frac{\tilde{w}(x)}{x}. \]

Hence
\[ r_{-1} = -(L + 1), \quad r_n = -\left(\tilde{\alpha}_n + \frac{\tilde{\beta}_n}{r_{n-1}}\right) \quad (n = 0, 1, \ldots). \] (37)

**Lemma 4.1.** The parameters $r_n$ have the explicit form
\[ r_n = -\frac{\psi_{n+1}}{\psi_{n+2}} \cdot \frac{L\psi_{n+2} + \xi \varphi_{n+2}}{L\psi_{n+1} + \xi \varphi_{n+1}} \quad (n \in \mathbb{N}_0). \] (38)

**Proof.** We will use the mathematical induction. For $n = 0$, we really get the expected value
\[ r_0 = -\frac{L^2 + 2L + 2}{(L + 1)(L + 2)}. \]
Suppose that it is true for $k = n$. Now, by the properties for $\varphi_n$ and $\psi_n$, we have
\[ \tilde{\alpha}_{n+1} \cdot r_n + \tilde{\beta}_{n+1} = -\frac{\psi_{n+1}}{\psi_{n+2}} \cdot \frac{L\psi_{n+2} + \xi \varphi_{n+2}}{L\psi_{n+1} + \xi \varphi_{n+1}}. \]
Dividing with $r_n$, we conclude that the formula is valid for $r_{n+1}$.

**Example 4.3.** For $L = 4$ we get
\[ r_{-1} = -5, \quad r_0 = -\frac{13}{15}, \quad r_1 = -\frac{51}{52}, \quad r_2 = -\frac{356}{357}, \]
wherefrom
\[ \alpha_0 = \frac{24}{5}, \quad \beta_0 = 5, \quad \alpha_1 = \frac{323}{65}, \quad \beta_1 = \frac{104}{25}, \quad \alpha_2 = \frac{1104}{221}, \quad \beta_2 = \frac{680}{169}, \]
just the same as in the Example 4.1.

**Proof of the main result.** Let us start from Krattenthaler formula
\[ h_1 = a_0, \quad h_n = a_0^n \beta_1^{n-1} \beta_2^{n-2} \cdots \beta_{n-2} \beta_{n-1} \quad (n = 2, 3, \ldots). \] (39)
Here is $a_0 = \beta_0 = L + 1$. This formula can be also written in the form
\[ h_1 = a_0, \quad h_n = \beta_0 \beta_1 \beta_2 \cdots \beta_{n-2} \beta_{n-1} \cdot h_{n-1}. \] (40)

From the theory of orthogonal polynomials, it is known that
\[ \|Q_{n-1}\|^2 = \beta_0 \beta_1 \beta_2 \cdots \beta_{n-2} \beta_{n-1} \quad (n = 2, 3, \ldots), \] (41)
wherefrom
\[ h_1 = a_0, \quad h_n = \|Q_{n-1}\|^2 \cdot h_{n-1} \quad (n = 2, 3, \ldots). \] (42)
Here,
\[ \|Q_{n-1}\|^2 = \beta_0 \frac{r_{n-2}}{r_{-1}} \prod_{k=0}^{n-2} \tilde{\beta}_k = \frac{L^{n-1}}{2} \cdot \frac{L\psi_{n} + \xi \varphi_{n}}{L\psi_{n-1} + \xi \varphi_{n-1}}. \] (43)
We will apply the mathematical induction again. The formula for $h_n$ is true for $n = 1$. Suppose that it is valid for $k = n - 1$. Then

$$h_n = \frac{L^{n-1}}{2} \cdot \frac{L \psi_n + \xi \varphi_n}{L \psi_{n-1} + \xi \varphi_{n-1}} \cdot \frac{L^{(n-1)(n-2)/2}}{2^n \xi} \cdot (L \psi_{n-1} + \xi \varphi_{n-1}),$$

wherefrom follows that the final statement

$$h_n = \frac{L^{n(n-1)/2}}{2^{n+1} \xi} \cdot (L \psi_n + \xi \varphi_n) \quad (n \in \mathbb{N})$$

is true.

REFERENCES


(Mentions sequences A005807, A001906, A001519.)

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