An iterative method for optimal resolution-constrained polar quantizer design

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Abstract

This paper addresses the problem of polar quantization optimization. Particularly, the aim of this investigation is to find the method for the optimal resolution-constrained polar quantizer design.

The new iterative algorithm for determination of the optimal decision and representation magnitude levels and algorithm for optimization of number of phase cells within each magnitude level, is proposed.

At high rates, the new optimal polar quantizer outperforms the optimal polar compander for 0.2dB, while the more significant gain should be expected at lower rates. In this paper, in order to enable practical implementation of quantizer model, algorithm which transforms real values for the optimal numbers of phase cells within magnitude levels into integer ones is also proposed. Moreover, the approximate closed form of signal to quantization ratio (SQNR) is derived.

Since circularly symmetric sources and complex presentation of signals arise in numerous applications, it can be concluded that the usage area of the suggested proposal is very wide (audio coding, image coding, spectral phase coding SPC, synthetic aperture radars systems SARs, coding of the discrete Fourier transform).

It should be emphasized that in contrast to earlier work, where models have been designed under high-rate assumption, the obtained nonuniform unrestricted polar quantizer is optimal for all rates.

1 Introduction

Many studies have considered the design of suboptimum vector quantizers that outperform the cartesian coordinate system quantizers, but with simpler implementation than optimal vector quantizers. In case of two-dimensional quantization of circularly symmetric densities, such implementation is the polar quantizer (Bucklew and Gallagher, 1979a, 1979b; Moo and Neuhoff, 1998; Pearlman, 1979; Perić and Stefanović, 2002; Swaszek, 1986; Swaszek and Ku, 1986; Swaszek and Thomas, 1983; Wilson, 1980). An intuitive reason for this superiority is that polar quantizers take advantage of the fact that circularly symmetric sources are characterized by contours of constant probability density function which are circles in the two-dimensional space (Bucklew and Gallagher, 1979a, 1979b; Jeong and Gibson, 1993; Pearlman, 1979; Perić and Stefanović, 2002; Perić et al., 2007; Swaszek and Ku, 1986; Wilson, 1980). Namely, polar quantizers have diverging angle separations, thus having small regions near the origin where the probability of vector occurrence is higher and enlarging the regions as they are removed from the origin. Hence, these schemes require the source symbols be represented in their polar form with the resulting polar coordinates processed by scalar quantizers (Bucklew and Gallagher, 1979a, 1979b; Moo and Neuhoff, 1998; Pearlman, 1979; Perić and Stefanović, 2002; Swaszek, 1986; Swaszek and Ku, 1986; Swaszek and Thomas, 1983; Wilson, 1980). According to the type of utilized scalar quantizers, there are several polar quantizer models. There are models in which uniform quantizers are applied for the phase as well as magnitude quantization (uniform polar quantizers) (Moo and Neuhoff, 1998; Perić and Stefanović, 2002; Swaszek, 1985). There are also models in which reconstruction and decision magnitude levels are not uniformly distributed i.e. nonuniform polar quantizers (Perić et al., 2007; Swaszek and Ku, 1986; Swaszek and Thomas, 1983). It should be also emphasized that Wilson (Wilson 1980) first defined the unrestricted polar quantizers, i.e. proposed a different number of level for the phase quantizers due to satisfaction of the mean square error criteria. Therefore, amplitude and phase can be quantized separately, which is in this case called strict polar quantization (SPQ) (Pearlman, 1979). They can be also quantized jointly when the phase quantization is made dependent on the amplitude. Such quantization is called unrestricted polar quantization (UPQ) (Wilson, 1980). Here can be noted that solution in (Wilson, 1980) is valid when the number of reconstruction points is small. The quantizers were derived analytically under high-rate assumptions. In (Swaszek and Thomas, 1983) the optimal nonuniform SPQ model is designed, while in (Swaszek, 1985) the idea of UPQ is realized for the large number of reconstruction points. In (Perić and Stefanović, 2002) the UPQ is optimized assuming that each scalar quantizer is a uniform one. In (Swaszek and Ku, 1986) the optimal UPQ is designed under constraint that scalar compander is used for magnitude processing. In the same paper, it is also shown that obtained optimal polar compander asymptotically approaches the optimal polar quantization performance. Opposite to cited researches, in this paper, during optimal polar quantuzer designing, the only constraint is a fixed number of reconstruction points (resolution constraint).

To derive the quantizer model which is optimal for all bitrates, we consider nonuniform UPQ model which does not engage scalar compander for the nonuniform distribution of the magnitude levels. Particularly, we extend the simple iterative algorithm presented in (Perić et al., 1998) and provide new one for the determination of the optimal reconstruction and decision magnitude levels. This enables us to overreach the asymptotic performance of the optimal polar compander (Swaszek and Ku, 1986) for 0.2dB. On the other side, we increase model complexity and processing delay, such that our model is more applicable for moderate and lower rates. We also provide an algorithm for the computation of optimal phase reconstruction points numbers in each magnitude region. Moreover, in order to enable simple performance calculation, we also derive the approximate expression for SQNR in closed form. The presented features assure that obtained solution should be of high significance, not only for researchers, but also for engineers. Taking into consideration that circularly symmetric sources and complex presentation of signals arise in numerous applications, it can be concluded that the usage area of this proposal is very wide.

Since short-time probability density function for speech signals is a well-modeled as Gaussian (Perić et al., 2007), polar quantization can be used for speech coding. Moreover, high performance and a simple control over perceptual effects of quantization motivate the usage of UPQ for sinusoidal audio coding (Pobloth et al., 2005; Popat and Zeger, 1992; Vafin and Kleijn, 2005). Since human eye sensitivity has circularly symmetric distribution, log-polar image sampling has been utilized recently (Boluda and Pardo, 2004; Metta et al., 2004; Shortt et al., 2006). Namely, in all applications where information is embedded in the phase or frequency of a carrier signal (for example, synthetic aperture radars systems SARs (Arslan, 2001; Perić and Jovković, 2002)), a polar analog-to-digital converter provides desirable phase information. As argued in (Pearlman and Gray, 1978), discrete Fourier transform of a fairly general data sources asymptotically leads to independent Fourier coefficients that have independent Gaussian real and imaginary parts. This means that polar quantization can be also used for compressed representation of coefficients obtained from the Fourier transform of signal (Pearlman and Gray, 1978; Pearlman, 1979). Nevertheless, taking into consideration that every source can be transformed in a Gaussian by means of properly chosen filtering technique (Popat and Zeger, 2007), polar quantizer model can also be applied to other sources. The practical significance of polar quantization is also illustrated through its involvement in two patents which are related to modulators and transmitters (Hasson and Barak, 2008; Zipper, 2008). In these patents polar quantization is used for signal constellations.

The remainder of this paper is organized as follows. In Section 2 polar quantization background is presented, while in Section 3 detailed analysis of the new iterative method for the optimal polar quantizer design is performed. Section 4 considers the iterative algorithm initialization. The achieved numerical results for bivariate Gaussian source are the topics addressed in Section 5. Finally, the summary and conclusions are provided in Section 6.

2 Polar quantization

We say that PDF $p_{X,Y}(x,y)$ of two dimensional random variable (X,Y) is *circular symmetric* if $p_{X,Y}(x,y) = g(\sqrt{x^2+y^2})$ for some function $g(r)$. In other words, $p_{X,Y}$ is circular symmetric if it is a function of only the radial component $r = \sqrt{x^2+y^2}$.

Two-dimensional vector quantizer $\mathcal{Q}$ is the function

$$\mathcal{Q} : \mathbb{R}^2 \to \{P_1, P_2, \ldots, P_N\} \subset \mathbb{R}^2$$

where $S_i = \mathcal{Q}^{-1}(P_i)$ is i-th *quantization cell* and $P_i \in \mathbb{R}^2$ is i-th *representation level* (*reconstruction point*). Polar coordinate system (r, θ) and polar quantizers are natural for two-dimensional data with circularly symmetric density. Assume that the plane $\mathbb{R}^2$ is partitioned into L *magnitude regions* by L *magnitude decision levels* (*region bounds*) $0 = r_1 < r_2 < \ldots < r_L < r_{L+1} = +\infty$. Furthermore assume that i-th amplitude region is divided into M_i *phase regions* where $M_i \geq 2$ is integer and $M_1 + M_2 + \ldots + M_L = N$. Value M_i is called *number of reconstruction points* in i-th region. It is assumed that *phase decision levels* $\theta_{i,j}$ are uniform, i.e. $\theta_{i,j} = 2(j-1)\pi/M_i$, for $j = 1, 2, \ldots, M_i$. This assumption is natural due to the circular symmetry of density function. Quantizer $\mathcal{Q}_{pol}$ is called *polar quantizer* if it cells $S_{i,j}$, $i = 1, 2, \ldots, L$ and $j = 1, 2, \ldots M_i$ are defined as

$$S_{i,j} = \{(r, \theta) \mid r_i \leq r < r_{i+1},\ \theta_{i,j} \leq \theta \leq \theta_{i,j+1}\}.$$

Assume that reconstruction points $P_{i,j}$ are given in the polar coordinates as (m_i, ψ_{ij}), i.e. that all have the same magnitude m_i for $j = 1, 2, \ldots, M_j$. Values m_i and $\psi_{i,j}$ are called *magnitude reconstruction level* and *phase reconstruction level respectively* and satisfy $r_i \leq m_i \leq r_{i+1}$ and $\psi_{i,j} = (2j-1)\pi/M_i$. These assumptions is also natural due to the circular symmetry of density function. In the rest of the paper we use the following vector notation: $\mathbf{r} = (r_2, r_3, \ldots, r_L)$, $\mathbf{m} = (m_1, m_2, \ldots, m_L)$ and $\mathbf{M} = (M_1, M_2, \ldots, M_L)$.

The quality of the quantizer $\mathcal{Q}$ is measured by *distortion* of resulting reproduction in comparison to the original. Mostly used measure of distortion is mean-squared error. It is defined as (Gersho and Gray, 1992)

$$D(\mathcal{Q}) = \sum_{i=1}^{N} \int_{S_i} d(P, P_i) p_{X,Y}(P) dP \tag{1}$$

where $d(\cdot, \cdot)$ is an Euclidean distance. The N-level quantizer $\mathcal{Q}^*$ is sad to be *optimal* if for any other N-level quantizer $\mathcal{Q}$ there holds $D(\mathcal{Q}) \geq D(\mathcal{Q}^*)$. In the case of polar quantizer $\mathcal{Q}_{pol}$, distorsion is given by (Swaszek and Ku, 1986; Perić et al., 2007)

$$D(\mathcal{Q}_{pol}) = \sum_{i=1}^{L} \sum_{j=1}^{M_i} \int_{r_i}^{r_{i+1}} \int_{\theta_{i,j}}^{\theta_{i,j+1}} (r^2 + m_i^2 - 2rm_i \cos(\theta - \psi_{i,j})) \frac{f(r)}{2\pi} dr d\theta. \tag{2}$$

where we denoted $f(r) = 2\pi r g(r)$. Similarly, we say that $\mathcal{Q}_{pol}^*$ is L-region *optimal polar quantizer* if $D(\mathcal{Q}_{pol}) \geq D(\mathcal{Q}_{pol}^*)$ for any other L-region polar quantizer $\mathcal{Q}_{pol}$ By solving the integral in (2) with respect to θ and using $\theta_{i,j} = 2(j-1)\pi/M_i$ and $\psi_{i,j} = (2j-1)\pi/M_i$ we obtain the following relation

$$D(\mathcal{Q}_{pol}) = D(\mathbf{r}; \mathbf{m}; \mathbf{M}) = \sum_{i=1}^{L} \int_{r_i}^{r_{i+1}} \left(r^2 + m_i^2 - 2rm_i \operatorname{sinc}\left(\frac{\pi}{M_i}\right)\right) f(r) dr. \tag{3}$$

Hence, the distorsion $D(\mathcal{Q})$ of the polar quantizer $\mathcal{Q}$ is represented as the function of the values r_i, m_i and M_i.

In the rest of the paper we assume that $\mathcal{Q}$ is given polar quantizer whose distorsion $D(\mathcal{Q})$ is given by relation (3). Our aim is to construct an iterative method for the computation of optimal polar quantizer $\mathcal{Q}^*_{pol}$.

3 Iterative method

An iterative method for optimization of two-dimensional polar quantizer, for circularly symmetric PDF, will be described. Let

$$F_0(a,b) = \int_a^b f(r)dr; \qquad F_1(a,b) = \int_a^b rf(r)dr. \tag{4}$$

Expression (3) can be written as

$$D(\mathcal{Q}_{pol}) = D(\mathbf{r};\mathbf{m};\mathbf{M}) = \sigma^2 + \sum_{i=1}^{L} m_i^2 F_0(r_i, r_{i+1}) - 2\sum_{i=1}^{L} m_i \operatorname{sinc}\left(\frac{\pi}{M_i}\right) F_1(r_i, r_{i+1}). \tag{5}$$

3.1 Optimization of r_i and m_i

Our goal is to minimize $D(\mathbf{r};\mathbf{m};\mathbf{M})$ under the constraints $0 = r_1 < r_2 < \ldots < r_L < r_{L+1} = +\infty$, $r_i \le m_i \le r_{i+1}$, $M_i \in \mathbb{N}$, $M_1 + M_2 + \ldots + M_L = N$. This is constrained, mixed non-linear optimization problem. For a fixed values of M_i, an optimal point $(\hat{\mathbf{r}}, \hat{\mathbf{m}})$ must satisfy the following conditions

$$\frac{\partial D}{\partial r_i} = \left(m_{i-1}^2 - m_i^2 - 2m_{i-1}r_i \operatorname{sinc}\left(\frac{\pi}{M_{i-1}}\right) + 2m_i r_i \operatorname{sinc}\left(\frac{\pi}{M_i}\right)\right) = 0, \tag{6}$$

$$\frac{\partial D}{\partial m_i} = 2m_i \int_{r_i}^{r_{i+1}} f(r)dr - 2\operatorname{sinc}\left(\frac{\pi}{M_i}\right)\int_{r_i}^{r_{i+1}} rf(r)dr = 0. \tag{7}$$

From the last equations we directly obtain

$$\hat{r}_i = \frac{\hat{m}_i^2 - \hat{m}_{i-1}^2}{2\left[\hat{m}_i \operatorname{sinc}\left(\frac{\pi}{M_i}\right) - \hat{m}_{i-1}\operatorname{sinc}\left(\frac{\pi}{M_{i-1}}\right)\right]}, \tag{8}$$

$$\hat{m}_i = \operatorname{sinc}\left(\frac{\pi}{M_i}\right)\frac{F_1(\hat{r}_i, \hat{r}_{i+1})}{F_0(\hat{r}_i, \hat{r}_{i+1})}. \tag{9}$$

Note that the condition (9) is similar to the centroid condition which is satisfied by representation levels of the optimal scalar quantizer. Also by direct calculation we find that

$$\left.\frac{\partial^2 D}{\partial r_i^2}\right|_{r_i=\hat{r}_i,\ m_i=\hat{m}_i} = (\hat{m}_i^2 - \hat{m}_{i-1}^2)\exp\left(-\frac{\hat{r}_i^2}{8}\right) > 0, \tag{10}$$

under the condition $m_i > m_{i-1}$. Since D is the quadratic function of m_i it is obvious that (9) gives its global minimum with respect to m_i. Above discussion confirms that (8) and (9) gives the global minimum of D, with respect to $\mathbf{r}$, if an optimal point $(\hat{\mathbf{r}}, \hat{\mathbf{m}})$ satisfies $\hat{r}_i \le \hat{m}_i \le \hat{r}_{i+1}$ for $i = 1, 2, \ldots, L$.

3.2 Optimization of M_i

Now consider the minimization of D with respect to $\mathbf{M}$. Recall that minimization is performed under the conditions $M_i \in \mathbb{N}$ and $M_1 + M_2 + \ldots + M_L = N$. Suppose that $\mathbf{r}$ is fixed and $\mathbf{m}$ is determined optimally, according to (8). Additionally suppose that $r_i < m_i < r_{i+1}$. By replacing (9) into (3) we obtain

$$D(\mathbf{r}; \mathbf{M}) = \int_0^{+\infty} r^2 f(r) dr - \sum_{i=1}^{L} A_i \operatorname{sinc}^2\left(\frac{\pi}{M_i}\right), \quad A_i = \frac{[F_1(\hat{r}_i, \hat{r}_{i+1})]^2}{F_0(\hat{r}_i, \hat{r}_{i+1})}, \tag{11}$$

We temporary replace the condition $M_i \in \mathbb{N}$ with the weaker $M_i \geq 1$. Since $2\sigma^2 = \int_0^{+\infty} r^2 f(r) dr$ is constant, the optimization problem reduces to

$$\begin{aligned} \max \quad & \sum_{i=1}^{L} A_i \operatorname{sinc}^2\left(\frac{\pi}{M_i}\right) \\ \text{s.t.} \quad & \sum_{i=1}^{L} M_i = N, \\ & M_i \geq 0. \end{aligned} \tag{12}$$

Last optimization problem can be solved using the Lagrange multipliers technique. Hence we construct Lagrange function $J(\mathbf{M})$ as

$$J(\mathbf{M}) = \sum_{i=1}^{L} A_i \operatorname{sinc}^2\left(\frac{\pi}{M_i}\right) - \lambda\left(\sum_{i=1}^{L} M_i - N\right), \tag{13}$$

and find its maximum under the conditions $M_i \geq 2$ (since at least two representation levels should be located in each amplitude region). By differentiating expression (13) we find the necessary conditions for the local minimum of the function J

$$\frac{\partial J}{\partial M_i} = a(M_i) - A_i \lambda = 0, \quad a(x) = \frac{x - x\cos\left(\frac{2\pi}{x}\right) - \pi \sin\left(\frac{2\pi}{x}\right)}{\pi^2}. \tag{14}$$

The following lemma proves that equation (system of equations) (14) has unique solution $\hat{M}_i$ for the fixed value of Lagrange multiplier λ.

Lemma 1 *Function $a(x)$ is monotonically decreasing, convex function on half-segment $[2, +\infty)$ i.e. $a'(x) < 0$ and $a''(x) > 0$, for every $x > 2$.*

Proof. First note that $a(x)$ is continuously differentiable function on the half-segment $[2, +\infty)$. Its derivative is given by

$$a'(x) = \frac{-x^2 \cos\left(\frac{2\pi}{x}\right) + x^2 - 2\pi \sin\left(\frac{2\pi}{x}\right) x + 2\pi^2 \cos\left(\frac{2\pi}{x}\right)}{\pi^2 x^2}. \tag{15}$$

Let

$$g(t) = a'\left(\frac{2\pi}{t}\right) = \frac{1}{\pi^2}\left[\frac{1}{2} t^2 \cos t - t \sin t - \cos t + 1\right]. \tag{16}$$

We consider function $g(t)$ on the half-segment $[0, \pi)$. It is also continuously differentiable on that segment. Since $g(0) = 0$ and $g'(t) = -\frac{1}{2\pi}t^2 \sin t < 0$ for $t \in [0, \pi)$ we conclude $g(t) < 0$ for $t \in (0, \pi)$ and $g(t)$ is strictly decreasing function on the same interval. Hence $a'(x) < 0$ for $x \in [2, \pi)$ and $a'(x)$ is strictly increasing function on the same half-interval. Last implies $a''(x) > 0$ and hence $a(x)$ is convex. $\square$

Since $a(x)$ is strictly decreasing function on $[2, +\infty)$ and $\lim_{x \to +\infty} a(x) = 0$, equation (14) has unique solution for the fixed value of Lagrange multiplier λ. Moreover, since

$$\frac{\partial^2 J}{\partial M_i^2} = a'(M_i) < 0, \quad \frac{\partial^2 J}{\partial M_i \partial M_j} = 0,$$

unique solution of the system (14) is the global maximum of the function $J(\mathbf{M})$ on the set $[2, \infty)^L$.

For a given value of Lagrange multiplier λ, denote by $\tilde{M}_i(\lambda)$ the unique solution of (14). Function $\tilde{M}_i(\lambda)$ is strictly decreasing, since $a(x)$ is strictly decreasing and obviously $A_i \geq 0$. Furthermore $\lim_{\lambda \to 0} \tilde{M}_i(\lambda) = +\infty$ since $\lim_{x \to +\infty} a(x) = 0$. Condition $M_i \geq 2$ implies that $\tilde{M}_i(\lambda)$ maps the half-segment $(0, \frac{0.4}{\lambda}]$ to $[2, +\infty)$.

Since the system (14) cannot be solved analytically, we obtain the approximate analytical expression for $\tilde{M}_i(\lambda)$. Function $a(x)$ can be expanded into the Taylor expansion around the point $x = +\infty$ as

$$a\left(\frac{1}{x}\right) = \frac{2\pi^2}{3x^3} + O\left(\frac{1}{x^5}\right). \tag{17}$$

By replacing (17) into (14) we obtain the following approximate solution

$$\hat{M}_i(\lambda) = \sqrt[3]{\frac{2\pi^2}{3\lambda A_i}}. \tag{18}$$

The absolute difference $|\hat{M}_i(\lambda) - \tilde{M}_i(\lambda)|$ can be bounded as follows

$$\begin{aligned} |\hat{M}_i(\lambda) - \tilde{M}_i(\lambda)| &= \tilde{M}_i(\lambda) \left| 1 - \sqrt[3]{\frac{2\pi^2}{3a(\tilde{M}_i(\lambda))}} \right| \\ &= \tilde{M}_i(\lambda) \left| 1 - \left(1 + \frac{4\pi^2}{45\tilde{M}_i^2(\lambda)} + O\left(\frac{1}{\tilde{M}_i^4(\lambda)}\right)\right)\right| \\ &= \frac{4\pi^2}{45\tilde{M}_i^2(\lambda)} + O\left(\frac{1}{\tilde{M}_i^3(\lambda)}\right). \end{aligned} \tag{19}$$

It is worth mentioning that $|\hat{M}_i(\lambda) - \tilde{M}_i(\lambda)| \to 0$ when $\tilde{M}_i(\lambda) \to 0$. It is also decreasing function (according to (19)) and for $\tilde{M}_i(\lambda) = 2$ there holds $|\hat{M}_i(\lambda) - 2| = 0.532$. Above discussions shows that the absolute error of the approximation of $\tilde{M}_i(\lambda)$ by $\hat{M}_i(\lambda)$ defined by (18) is less than 1 whenever $\tilde{M}_i(\lambda) \geq 2$.

The value of Lagrange multiplier can be found from the condition $\sum_{i=1}^{L} \hat{M}_1(\lambda) = N$. Using the relation (18) we directly found

$$\lambda = \frac{2\pi^2}{3N^3} \left(\sum_{i=1}^{L} A_i^{-1/3} \right)^3 . \tag{20}$$

By replacing (20) in (18) we obtain the following approximate solution of the optimization problem (12):

$$\hat{M}_i = N \frac{A_i^{-1/3}}{\sum_{j=1}^{L} A_j^{-1/3}} \tag{21}$$

Optimization problem (12) is an integer programming (IP) problem, since M_i are integers. However the solution given by (21) is not an integer, in general. Hence we round $\hat{M}_i$ to the closest integer value, i.e. we set $M_i^* = \mathrm{round}(\hat{M}_i)$. However after the rounding operation, values M_i^* might not satisfy the condition $\sum_{i=1}^{L} M_i^* = N$. If the sum on the left side is larger than N, the difference $\delta = \sum_{i=1}^{L} M_i^* = N$ is substracted from M_i^* corresponding to the smallest value A_i. Similarly, if the sum is less than N, difference is added to the M_i^* corresponding to the largest A_i.

3.3 Algorithm for optimizing M_i

Since the optimization problem (12) is nonlinear integer programming problem, we can apply the conventional techniques for its solving (for example, Branch and Bound method (Li and Sun, 2006)). The point $\mathbf{M}^* = (M_1^*, M_2^*, \ldots, M_L^*)$ can be used as a initial point. However, this approach requires the implementation of integer programming method (or using the IP solver, for example MOSEK, CPLEX, etc.). We present another method which is simple for implementation and gives the results close to the optimal.

Values M_i^* can be furthermore improved by the following procedure. Pick the indices i and j such that $i < j$. Denote

$$f_{ij}(x) = A_i \,\mathrm{sinc}^2\left(\frac{\pi}{x}\right) + A_j \,\mathrm{sinc}^2\left(\frac{\pi}{M_i^* + M_j^* - x}\right).$$

Note that $f_{ij}(M_i^*)$ is the sum of two summands corresponding to M_i^* and M_j^* in the objective function from the optimization problem (12). From Lemma 1 we have

$$f_{ij}''(x) = A_i a'(x) + A_j a'(M_i^* + M_j^* - x) < 0$$

for every $2 \le x \le M_i^* + M_j^* - 2$. Since $f_{ij}''(x) < 0$, function $f_{ij}(x)$ has at most one global minimum in the segment $[2, M_{ij} - 2]$ where $M_{ij} = M_i^* + M_j^*$. This minimum can be computed as the unique solution $\hat{x}$ of the equation $f_{ij}'(x) = 0$, if $f_{ij}'(2) f_{ij}'(M_{ij} - 2) < 0$. Else it is one of the boundary points of the segment $[2, M_{ij} - 2]$ (this case never happens in practice, but it is possible theoretically). Such obtained value $\hat{x}$ is again not integer, in general and hence we have to apply the rounding again. However since $f''_{ij}(x) < 0$, integer maximum x^* is one of the values $\lfloor \hat{x} \rfloor$ and $\lceil \hat{x} \rceil$. Value M_i^* can be used as a starting point. After the procedure value M_i^* is set to x^* and M_j^* is set to $M_{ij} - x^*$.

Since we need an integer maximum of $f_{ij}(x)$ and M_i^* is a good approximation, we can obtain x^* simply by incrementing or decrementing M_i^* while f_{ij} increases. More strictly, it is realised by the following procedure:

Algorithm 1 Imp(A_i, A_j, M_i^*, M_j^*) - Integer maximization of $f_{ij}(x)$

Require: Integers $M_i^*, M_j^* \geq 2$ and positive reals A_i, A_j.

1: $M_{ij} := M_i^* + M_j^*$
2: $x := M_i^*$
3: **while** ($f_{ij}(x)$ is increasing) and ($x < M_{ij} - 2$) **do**
4: $\quad x := x + 1$
5: **end while**
6: **while** ($f_{ij}(x)$ is increasing) and ($x > 2$) **do**
7: $\quad x := x - 1$
8: **end while**
9: **return** x

Note that exactly one of the while loops in steps 2 and 5 will be accessed. Algorithm 1 can be applied for each pair of indices $\{i, j\}$. In the practice, initial value of M_i^* is usually very good approximation of the . Hence the number of steps of the Algorithm 1 is less than 3. When applied to each pair $\{i, j\}$, Algorithm 1 improves the initial point in only few number of pairs. All above discussion approves that initial values of M_i^* are excellent approximation of the optimal solution of the optimization problem (12). Complete procedure is summarized in the Algorithm 2.

3.4 Algorithm for the iterative method

Now we are ready to formulate the complete iterative method for design of the optimal polar quantizer for circular symmetric source density. Initial values are:

1. Number of region boundaries L,
2. Total number of reconstruction levels N,
3. Initial region boundaries $0 = r_1^0 < r_2^0 < \cdots < r_L^0 < r_{L+1}^0 = +\infty$.

In each step, algorithm first computes the new values M_i^k by Algorithm 2 and known values r_i^{k-1}. Then new values r_i^k are computed by relation (8) and known values m_i^{k-1} and M_i^k. Finally, the new magnitude reconstruction levels m_i^k are computed by relation (9) and known values r_i^k and M_i^k. Algorithm is terminated when the relative difference between the distorsion in k-th and $k-1$-th iteration is less than ϵ.

Algorithm 2 $\mathbf{OptM}(N;\mathbf{A})$ - Optimization of the numbers of reconstruction levels in regions

Require: Values $\mathbf{A} = (A_1, A_2, \dots, A_L)$

$\hat{M}_i := N \dfrac{A_i^{-1/3}}{\sum_{j=1}^{L} A_j^{-1/3}}$, for $i = 1, 2, \dots, L$
$M_i^* := \mathrm{round}(\hat{M}_i)$, for $i = 1, 2, \dots, L$
$\delta := \sum_{i=1}^{L} M_i^* - N$
if $\delta > 0$ **then**
    Let l be index such that A_l is minimal
    $M_l^* := M_l^* - \delta$
else
    if $\delta < 0$ **then**
        Let l be index such that A_l is maximal
        $M_l^* := M_l^* + |\delta|$
    end if
end if
for each pair $\{i, j\}$, $1 \le i < j \le L$ **do**
    $M_i^* := \mathbf{Imp}(A_i, A_j, M_i^*, M_j^*)$
end for
return $\mathbf{M}^* = (M_1^*, M_2^*, \dots, M_L^*)$

Above procedure have one drawback. It is not guaranteed that values r_i^k, computed by relation (9) satisfy

$$0 = r_1^k < r_2^k < \cdots < r_L^k < r_{L+1}^k = +\infty. \tag{22}$$

We say that such vector $\mathbf{r}$ is *degenerate*. For sufficiently high values M_i^k, we have $\mathrm{sinc}(\pi/M_i^k) \approx 1$ and according to (9) holds $r_i^k = (m_{i-1}^{k-1} + m_i^{k-1})/2$. In such case, if m_i^{k-1} are in ascending order, the same holds for r_i^k. So, $\mathbf{r}$ is not degenerate. But if M_i^k are not sufficiently high (which is usually the case for $i = 1, 2$ where M_i^k is around 4), relation (22) may not hold. Let p be the minimal index such that r_p^k violates the ascending order condition. Recall that, according to equation (10), value r_p^k is global minimum of D as the function of r_p, under the condition that $M_j = M_j^k$ are fixed, $m_j = m_j^{k-1}$ are fixed and in ascending order. Since $r_p^k < r_{p-1}^k$, $r_p = r_{p-1}^k$ is minimum of D, as the function r_p, under the condition $r_p > r_{p-1}^k$. Hence, $p-1$-th region (between r_{p-1}^k and r_p^k) vanishes. Hence, the value r_p^k should be dropped, M_{p-1}^k points from $p-1$-th region should be assigned to the p-th region (i.e. $M_p^k := M_p^k + M_{p-1}^k$). By repeating above procedure, we can eliminate all values r_i^k violating ascending condition. Note that last procedure does not increase the value of distorsion D. Algorithm 3 removes the degeneracy of $\mathbf{r}$ and is based on above discussion.

Now we are ready to formulate Algorithm 4 for iterative construction of the optimal polar quantizer for circular symmetric source density.

In practice, degeneracy occurs very rarely, when the initial conditions for Algorithm 4 are suitably chosen. In next section we show one way for choosing an initial conditions and number of regions L.

Algorithm 3 $\mathbf{FixAsc}(\mathbf{r}, \mathbf{M}, L)$ - removing the degeneracy of $\mathbf{r}$

Require: Sequence of region bounds $\mathbf{r} = (r_1, r_2, \ldots, r_L, r_{L+1})$ where $r_1 = 0$, $r_{L+1} = +\infty$, sequence of reconstruction points numbers in each region $\mathbf{M} = (M_1, M_2, \ldots, M_L)$ and number of regions L.

$p := 1$;
while $(p \leq L)$ **do**
  if $r_p < r_{p-1}$ **then**
    $M_{p+1} := M_{p+1} + M_p$
    $r_{j+1} := r_j$, $M_{j+1} := M_j$, for $j = p, p+1, \ldots, L$
    $L := L - 1$
  else
    $p := p + 1$
  end if
end while
return $(\mathbf{r}, \mathbf{M}, L)$

Algorithm 4 $\mathbf{OptPQ}(L, N, \mathbf{r}^0, \epsilon)$ - Construction of the optimal polar quantizer for circular symmetric source density

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Require: Number of regions L and reconstruction points N, initial values r_i^0 and M_i^0 such that $0 = r_1^0 < r_2^0 < \cdots < r_L^0 < r_{L+1}^0 = +\infty$ and $\sum_{i=1}^{L} M_i^0 = N$ and precision ϵ.

$k := 1$
$D^1 := +\infty$
repeat
  $k := k + 1$
  $A_i^k := \dfrac{[F_1(r_i^{k-1}, r_{i+1}^{k-1})]^2}{F_0(r_i^{k-1}, r_{i+1}^{k-1})}$, for $i = 1, 2, \ldots, L$
  $\mathbf{M}^k := \mathbf{OptM}(N, \mathbf{A}^k)$
  $r_i^k := \dfrac{(m_i^{k-1})^2 - (m_{i-1}^{k-1})^2}{2\left[m_i^{k-1} \operatorname{sinc}\left(\frac{\pi}{M_i^k}\right) - m_{i-1}^{k-1} \operatorname{sinc}\left(\frac{\pi}{M_{i-1}^k}\right)\right]}$, for $i = 2, 3, \ldots, L$ $(r_1^k := 0, r_{L+1}^k := +\infty)$
  $(\mathbf{r}^k, \mathbf{M}^k, L) := \mathbf{FixAsc}(\mathbf{r}^k, \mathbf{M}^k, L)$
  $m_i^k := \operatorname{sinc}\left(\frac{\pi}{M_i^k}\right) \dfrac{F_1(r_i^k, r_{i+1}^k)}{F_0(r_i^k, r_{i+1}^k)}$, for $i = 1, 2, \ldots, L$
  Compute $D^k := D(\mathbf{r}^k, \mathbf{m}^k, \mathbf{M}^k)$ using the relation (5).
until $(D^{k-1} - D^k)/D^k < \epsilon$
return $(\mathbf{r}^k, \mathbf{m}^k, \mathbf{M}^k)$

On the other side, Algorithm 4 is always convergent, since the distorsion D^k always decreases. However, as in the case of the Lloyd-Max algorithm for scalar quantizers (Max, 1960), it is not guaranteed that the solution obtained by Algorithm 4 is optimal.

4 Initial values

This section provide an efficient way to choose initial values $\mathbf{r}^0$ and L for the Algorithm 4. It is based on the result of Swaszek and Ku (Swaszek and Ku, 1986). This approach is based on the *companding technique* and provide the an asymptotically optimal quantizer. Quantizers obtained using companding technique are called *companding quantizers.* This technique is applicable for various types of quantizers, see for example (Gersho and Gray, 1992; Jayant and Noll, 1984; Perić et al., 2007; Swaszek and Ku, 1986).

An companding polar quantizer $\mathcal{Q}_{pol,com}(x)$ is defined as $\mathcal{Q}_{pol,com}(x) = G^{-1}(\mathcal{UQ}_{pol}(G(x)))$ where $\mathcal{UQ}_{pol}(x)$ is an uniform polar quantizer and $G(x)$ is *compressor function.* It is defined by

$$G : \mathbb{R}^2 \mapsto B(\mathbf{0}, 1), \qquad G(x) = \frac{g(r)}{r} x, \quad r = \|x\|.$$

where $g : \mathbb{R} \mapsto [0, 1]$ is *polar compressor function* and $B(\mathbf{0}, 1)$ is unit ball. The region bounds y_i and magnitude representation levels z_i of the L-region *uniform polar quantizer* $\mathcal{UQ}_{pol}(x)$ are given respectively by

$$y_i = \frac{i-1}{L}, \; i = 1, 2, \ldots, L+1, \qquad z_i = \frac{y_i + y_{i+1}}{2} = \frac{2i-1}{2L}, \; i = 1, 2, \ldots, L. \tag{23}$$

Region bounds r_i an magnitude representation levels m_i of companding polar quantizer $\mathcal{Q}_{pol,com}(x)$ are given by $r_i = g^{-1}(y_i)$ and $m_i = g^{-1}(z_i)$.

Swaszek and Ku considered an optimization of the companding polar quantizer $\mathcal{Q}_{pol,com}(x)$. According to (Swaszek and Ku, 1986), optimal number of regions L and number of reconstruction points in i-th region are given by

$$L = \operatorname{round}\left[\frac{\sqrt{N}}{\sqrt{2\pi}} \frac{\int_0^{+\infty} x^{-1/4} f^{1/4}(x)dx}{\left[\int_0^{+\infty} x^{-1/2} f^{1/2}(x)dx\right]^{1/2}}\right] \tag{24}$$

$$M_i^0 = \frac{\sqrt{2\pi N} f^{1/4}(m_i) m_i^{3/4}}{\left[\int_0^{+\infty} x^{-1/2} f^{1/2}(x)dx\right]^{1/2}}, \tag{25}$$

while an optimal polar compressor function $g(r)$ is defined by

$$g(r) = \frac{\int_0^r x^{-1/4} f^{1/4}(x)dx}{\int_0^{+\infty} x^{-1/4} f^{1/4}(x)dx}. \tag{26}$$

Recall that $r_i = g^{-1}(y_i)$ and $m_i = g^{-1}(z_i)$. Since L is integer, it is given as the integer closest to the expression in (24). However, it should be checked also values $L - 1$ and $L + 1$.

Since the optimal companding polar quantizer is asymptotically optimal polar quantizer (Swaszek and Ku, 1986), it can be used for the start of Algorithm 4. In other words, initial parameters of Algorithm 4 can be chosen as the corresponding parameters of optimal companding polar quantizer. In such way, number of regions should be chosen according to (24) (also values $L-1$ and $L+1$ should be tried) and initial region bounds $r_i^0 = g^{-1}(y_i)$ where $g(r)$ is given by (26).

5 Two-dimensional Gaussian source and numerical examples

We test our algorithms on designing the optimal polar quantizer for two-dimensional Gaussian source. Let (X,Y) be two-dimensional Gaussian random variable such that X and Y are uncorrelated. PDF function $p_{X,Y}(x,y)$ and function $f(r)$ are given by

$$p_{X,Y}(x,y) = \frac{1}{2\pi\sigma}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right), \quad f(r) = r\exp\left(-\frac{r^2}{2}\right). \tag{27}$$

By direct computation using (4) we find

$$\begin{aligned} F_1(a,b) &= \exp\left(-\frac{a^2}{2}\right) - \exp\left(-\frac{b^2}{2}\right), \\ F_2(a,b) &= a\exp\left(-\frac{a^2}{2}\right) - \sqrt{\frac{\pi}{2}}\mathrm{erf}\left(\frac{a}{\sqrt{2}}\right) - b\exp\left(-\frac{b^2}{2}\right) + \sqrt{\frac{\pi}{2}}\mathrm{erf}\left(\frac{b}{\sqrt{2}}\right) \end{aligned} \tag{28}$$

where $\mathrm{erf}(x) = 2\pi^{-1/2}\int_0^x \exp(-t^2)dt$ is an error function.

All algorithms are implemented in the symbolic programming package Mathematica (version 7.0). We show some results obtained by testing our implementations for uncorrelated two-variable Gaussian source with unit variance ($\sigma = 1$). Parameters of the optimal polar quantizer for $N = 64$ levels are given by

$$\begin{aligned} L &= 6 \\ \mathbf{r} &= (0, 0.276903, 0.663063, 1.11976, 1.65484, 2.34537, +\infty) \\ \mathbf{m} &= (0.11707, 0.46865, 0.883483, 1.36178, 1.93031, 2.69542) \\ \mathbf{M} &= (2, 6, 11, 15, 16, 14) \\ D &= 0.0620599, \quad \mathrm{SQNR} = 15.0822 \text{ dB}. \end{aligned}$$

Also, parameters of the optimal polar quantizer for $N = 128$ levels are given by

$$\begin{aligned} L &= 8 \\ \mathbf{r} &= (0, 0.323755, 0.62253, 0.943175, 1.29048, 1.68843, 2.16081, 2.80557, +\infty) \\ \mathbf{m} &= (0.193301, 0.475613, 0.780552, 1.10901, 1.47377, 1.89305, 2.4061, 3.10778) \\ \mathbf{M} &= (4, 9, 14, 18, 22, 23, 22, 16) \\ D &= 0.0317631, \quad \mathrm{SQNR} = 17.9911 \text{ dB}. \end{aligned}$$

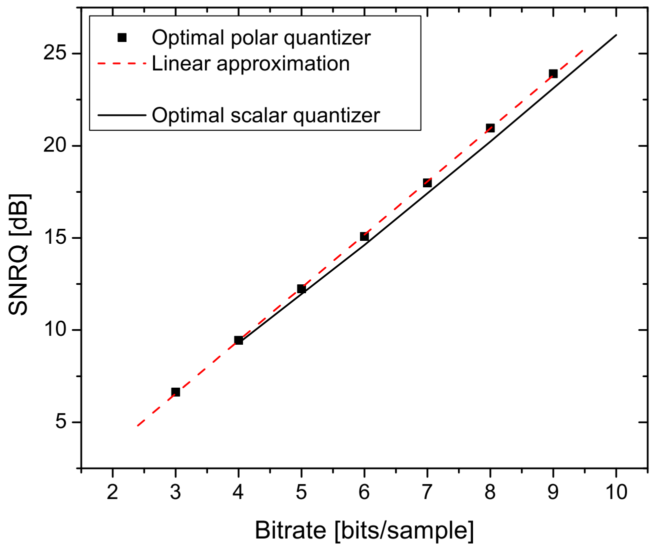

Figure 1: SQNR value of optimal polar quantizer and optimal scalar quantizer for different values of bitrate R.

Here D denotes the distorsion of quantizer, defined by (3) and $\mathrm{SQNR} = 10\log(2\sigma^2/D)$ denotes the value of Signal-to-Quantizer-Noise-Ratio. Figure 1 shows the dependence of SQNR of the optimal polar quantizer versus the total bitrate $R = \log_2 N$. We also included, for the purpose of comparation, the dependence of SQNR for the optimal scalar quantizer (Jayant and Noll, 1984).

It can be noticed that the dependence is almost linear. Asymptotic expression for the distorsion D, valid for large values of N, is equal to $D = \frac{2\pi}{3N}$ (Perić, et al., 2007; Swaszek and Ku, 1986). Therefore, the asymptotic dependence of SQNR, as a function of R is also linear. By linear regression we obtain the approximate values of the parameters of linear dependence $\mathrm{SQNR} \approx -2.09141 + 2.87887 \cdot R$, from Figure 1, where the correlation coefficient equal to 0.99992. This confirms the linear dependence of SQNR for smaller values of bitrate R.

Note that for $N = 256$, the distorsion and SQNR of the optimal polar quantizer are given by $D = 0.0159589$ and $\mathrm{SQNR} = 20.9072$ dB. Optimal uniform polar quantizer (UPQ) (Swaszek and Ku, 1986) has total distorsion $D_{UQ} = 0.01683$ and $\mathrm{SQNR} = 20.7495$ dB while an optimal two-dimensional vector quantizer has $D_{VQ} = 0.01575$ and $\mathrm{SQNR} = 21.0375$ dB (Gersho and Gray, 1992). Hence the difference in SQNR values between an optimal vector and optimal polar quantizer is only 0.057224 dB while the difference between an optimal vector and optimal uniform polar quantizer is 0.28 dB.

Number of iterations required for our method (Algorithm 4), for $\epsilon = 10^{-6}$ is 244. It shows the slow convergence of our method.

6 Conclusion

In this paper we present one new method for the resolution-constrained polar quantization optimization. We provide iterative algorithm for determination of the optimal reconstruction and decision magnitude levels, as well as, algorithm for optimization of number of phase cells within each magnitude level. We point out that firstly we obtain real values for the optimal numbers of reconstruction points in magnitude regions and after that we assure algorithm which enables transformation of optimal real values to the optimal integer ones.

The concept of proposed design method can be also considered as the iterative improving of the optimal polar compander. The achieved gain in reproduced signal quality is not smaller than 0.2dB, while design and implementation complexities are enlarged. Because of that, the obtained optimal polar quantizer can be used for moderate and lower rates in analog-to-digital conversion of signals with circularly symmetric densities and complex presentation (audio coding, image coding, spectral phase coding SPC, synthetic aperture radars systems SARs, coding of the discrete Fourier transform). Furthermore, a possibility that any kind of density can be transformed in a Gaussian distribution by means of properly chosen filtering gives additional importance to the proposed quantizer model.

In this paper we also derive the approximate expression for SQNR in closed form which enables easily calculation of optimal polar quantizer performances. Therefore, we believe that the novel quantizer model is of high significance not only for researchers but also for engineers.

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