Modern Functional Analysis in the Theory of Sequence Spaces and Matrix Transformations

1st International Conference on Mathematical Sciences
Gaza, Palestine, 15–17 May, 2006

Eberhard Malkowsky

Mathematisches Institut
Justus–Liebig Universität Gießen
Arndtstraße 2
D-35392 Gießen
Germany

Department of Mathematics
Faculty of Sciences
and Mathematics
University of Niš, Višegradska 33
18000 Niš, Serbia and Montenegro

email: eberhard.malkowsky@math.uni-giessen.de
ema@bankerinter.net; ema@pmf.ni.ac.yu

[1]Research supported by the research grants #1232 and #144003 of the Serbian Ministry of Science, Technology and Development
AMS Subject Classification 2000: Primary: 40H05; Secondary: 46A15

1. Introduction

Several concepts and theories in functional analysis have turned out to be powerful and widely used tools in operator theory, in particular in the theory of matrix transformations in summability.

We study the theories of

- FK, BK, AK and AD spaces
- multiplier and dual spaces
- matrix transformations
- measures of noncompactness

Summability

The classical summability theory deals with a generalisation of the convergence of sequences or series of real or complex numbers. The idea is to assign a limit to divergent sequences or series by considering a transform. Most popular are matrix transformations.

Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix, and $x = (x_k)_{k=0}^{\infty}$ be a sequence of complex numbers. Then A defines a matrix transformation or summability method A by

$$y_n = (Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k \text{ for } n = 0, 1, \ldots. \tag{1.1}$$

The sequence $x = (x_k)_{k=0}^{\infty}$ is said to be summable A to ℓ, if

$$\lim_{n \to \infty} y_n = \ell \text{ exists.}$$

The most important summability methods are given by

- Hausdorff matrices and their special cases
  - Cesàro
  - Euler
  - Hölder matrices
- Nörlund matrices

We refer to [Har, Mad, Pey, Z–B] for the classical summability theory.

Matrix Transformations

The theory of matrix transformations deals with establishing necessary and sufficient conditions on the entries of a matrix to map a sequence space X into a sequence space Y.

Let ω, c and ℓ_∞ denote the sets of all complex, convergent and bounded sequences. Given X,$Y \subset \omega$, we write (X, Y) for the class of all infinite matrices that map X into Y. So $A \in (X, Y)$ if and only if the series $(Ax)_n$ in (1.1) converge for all n and all $x \in X$ and

$$Ax = (Ax)_{n=0}^{\infty} \in Y \text{ for all } x \in X. \tag{1.2}$$

The first results were the Toeplitz theorem for the class (c, c), the characterisation of conservative matrices, and the Schur theorem for the class (ℓ_∞, c), the characterisation of coercive matrices.

Theorem 1.1 (O. Toeplitz, 1911) ([Toe]) $A \in (c, c)$ if and only if

(i)
$$\|A\| = \sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty,$$

(ii)
$$\lim_{n\to\infty} a_{nk} = \alpha_k \text{ exists for every } k$$

and

(iii)
$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = \alpha \text{ exists.}$$

Theorem 1.2 (O. Schur, 1920) $A \in (\ell_\infty, c)$ *if and only if*

(i)
$$\sup_n \sum_{k=0}^{\infty} |a_{nk}| \textit{ is uniformly convergent in } n$$

and

(ii)
$$\lim_{n\to\infty} a_{nk} = \alpha_k \textit{ exists for every } k.$$

Applications

Example **1.3** Steinhaus type theorems

A matrix is called regular if it is conservative and preserves limits. Toeplitz also proved that A is conservative if and only if conditions (i), (ii) and (iii) of Theorem 1.1 hold with $\alpha_k = 0$ for all k and $\alpha = 1$.

The Steinhaus theorem states that, for every regular matrix A, there is a bounded sequence which is not summable A.

Proof. Assume there is a regular matrix $A \in (\ell_\infty, c)$. Then it follows from Theorem 1.1 (iii), (ii), and Theorem 1.2 (i) that

$$1 = \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = \sum_{k=0}^{\infty} \left(\lim_{n\to\infty} a_{nk} \right) = 0.$$

□

Example **1.4** *Weak and strong convergence coincide in ℓ_1, the set of all absolutely convergent series.*

We assume that the sequence $(x^{(n)})_{n=0}^{\infty}$ is weakly convergent to x in ℓ_1, that is

$$f(x^{(n)}) - f(x) \to 0 \text{ for every } f \in \ell_1^*.$$

Since ℓ_1^* and ℓ_∞ are norm isomorphic, to every $f \in \ell_1^*$ there corresponds a sequence $a \in \ell_\infty$ such that

$$f(y) = \sum_{k=0}^{\infty} a_k y_k \text{ for all } y \in \ell_1.$$

We define the matrix $B = (b_{nk})_{n,k=0}^{\infty}$ by $b_{nk} = x_k^{(n)} - x_k \ (n, k = 0, 1, \dots)$. Then we have for all $a \in \ell_\infty$

$$f(x^{(n)}) - f(x) = \sum_{k=0}^{\infty} a_k \left(x_k^{(n)} - x_k \right) = \sum_{k=0}^{\infty} b_{nk} a_k \to 0 \ (n \to \infty),$$

that is $B \in (\ell_\infty, c)$. It follows from Theorem 1.2 that

$$\|x^{(n)} - x\|_1 = \sum_{k=0}^{\infty} |x_k^{(n)} - x_k| = \sum_{k=0}^{\infty} |b_{nk}| \to 0 \ (n \to \infty).$$

Surveys of results on matrix transformations can be found in [S–T, Z–B, Wil2, Mad, M–R], and in [Mad1] for infinite matrices of operators.

2. FK, BK, AK and AD Spaces

The theory of FK spaces is the most powerful tool in the theory of matrix transformations ([Wil1, Wil2, K–G, Zel, M–R]).

Definition 2.1 *Let H be a linear space and a Hausdorff space. An* **FH space** *is a (locally convex) Fréchet space X such that X is a linear subspace of H and the topology of X is stronger than the restriction of the topology of H on X.*

If $H = \omega$ with its topology given by the metric d with

$$d(x, y) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{|x_k - y_k|}{1 + |x_k - y_k|} \; (x, y \in \omega), \tag{2.1}$$

then an FH space is called an **FK space**.

A **BH space** *or a* **BK space** *is an FH or FK space which is a Banach space.*

Remark 2.2 *Since convergence in (ω, d) and coordinatewise convergence are equivalent, convergence in an FK space implies coordinatewise convergence.*

Example **2.3** *Let $H = \mathcal{F} = \{f : [0,1] \to \mathbb{R}\}$, and, for every $t \in [0,1]$, let $\hat{t} : \mathcal{F} \to \mathbb{R}$ be the function with $\hat{t}(f) = t(f)$. We assume that $\mathcal{F}$ has the weak topology by $\Phi = \{\hat{t} : t \in [0,1]\}$. Then $C[0,1]$ is a BH space with $\|f\| = \sup_{t\in[0,1]} |f(t)|$.*

Proof. Let $(f_k)_{k=0}^{\infty}$ be a sequence in $C[0,1]$ with $f_k \to 0$ $(k \to \infty)$, then

$$\hat{t}(f_k) = f_k(t) \to 0 \ (k \to \infty) \text{ for all } \hat{t} \in \Phi,$$

that is $f_k \to 0$ $(k \to \infty)$ in $\mathcal{F}$. □

Example **2.4** *Trivially ω is an FK space with its metric defined in (2.1)*

The sets ℓ_∞, c and c_0 (of null sequences), and ℓ_1 are Banach spaces with the natural norms

$$\|x\|_\infty = \sup |x_k| \text{ on } \ell_\infty, c, c_0$$

and

$$\|x\| = \sum_{k=0}^{\infty} |x_k| \text{ on } \ell_1.$$

Since

$$x_k \leq \|x\| \text{ in each case,}$$

norm convergence implies coordinatewise convergence.

So these spaces are BK spaces.

Theorem 2.5 *Let X be a Fréchet space, Y be an FH space and $f : X \to Y$ be linear.*
If $f : X \to H$ is continuous, then $f : X \to Y$ is continuous.

Proof. Let $\mathcal{T}_X$, $\mathcal{T}_Y$ and $\mathcal{T}_H$ be the topologies on X, Y and of H on Y.
If $f : X \to (Y, \mathcal{T}_H)$ is continuous, then it has closed graph by the closed graph lemma (any continuous map to a Hausdorff space has closed graph).
Since Y is an FH space, we have $\mathcal{T}_H \subset \mathcal{T}_Y$, and so $f : X \to (Y, \mathcal{T}_Y)$ has closed graph.
Consequently $f : X \to (Y, \mathcal{T}_Y)$ is continuous by the closed graph theorem.
□

Corollary 2.6 *Let X be a Fréchet space, Y be an FK space, $f : X \to Y$ be linear, and $P_n : X \to \mathbb{C}$ $(n \in \mathbb{N}_0)$ be defined by $P_n(x) = x_n$.*
If $P_n \circ f : X \to \mathbb{C}$ is continuous for every n, then $f : X \to Y$ is continuous.

Proof. Since convergence and coordinatewise convergence are equivalent in ω, the continuity of $P_n : X \to \mathbb{C}$ for all n implies the continuity of $f : X \to \omega$, hence of $f : X \to Y$ by Theorem 2.5. $\square$

Theorem 2.7 *Let $X \supset \phi$ be an FK space where ϕ denotes the set of all finite sequences.*
If the series $\sum_{k=0}^{\infty} a_k x_k$ converges for all $x \in X$, then the linear functional $f_a : X \to \mathbb{C}$ defined by

$$f_a(x) = \sum_{k=0}^{\infty} a_k x_k \text{ for all } x \in X$$

is continuous.

Proof. We define $f_a^{[n]} : X \to \mathbb{C}$ $(n \in \mathbb{N}_0)$ by

$$f_a^{[n]}(x) = \sum_{k=0}^{n} a_k x_k \text{ for all } x \in X.$$

Since X is an FK space, $f_a^{[n]} \in X^*$ for all n. The limit $f_a(x) = \lim_{n\to\infty} f_a^{[n]}(x)$ exists for all $x \in X$, hence $f_a \in X^*$ by the Banach–Steinhaus theorem. □

Theorem 2.8 *Any matrix map between FK spaces is continuous.*

Proof. Let X and Y be FK spaces, $A \in (X, Y)$ and $f_A : X \to Y$ be defined by $f_A(x) = Ax$ for all $x \in X$.
Since the maps $P_n \circ f_A : X \to \mathbb{C}$ are continuous for all n by Theorem 2.7, $f_A : X \to Y$ is continous by Corollary 2.6. □

The FH topology of an FH space is unique.

Theorem 2.9 *Let X and Y be FH spaces with $X \subset Y$.*
Then the topolgy $\mathcal{T}_X$ is larger than the topology $\mathcal{T}_Y|_X$ of Y on X.
They are equal if and only if X is a closed subspace of Y.
In particular, the topology of an FH space is unique.

Proof. Apply Theorem 2.5 to the inclusion map $\iota : X \to Y$ to obtain all the statements except that about the equality of the topologies.
If X is closed in Y then X becomes an FH space with $\mathcal{T}_Y|_X$. By the uniqueness $\mathcal{T}_X = \mathcal{T}_Y|_X$.
If $\mathcal{T}_X = \mathcal{T}_Y|_X$, then X is a complete, hence closed, subspace of Y. $\square$

The class of FK spaces is fairly large.

Example **2.10** A Banach sequence space which is not a BK space

We consider the spaces $(c_0, \|\cdot\|_\infty)$ *and*

$$\ell_2 = \left\{ x \in \omega : \sum_{k=0}^{\infty} |x_k|^2 < \infty \right\} \text{ with } \|x\|_2 = \left(\sum_{k=0}^{\infty} |x_k|^2 \right)^{1/2}.$$

Since they have the same algebraic dimension, there is an isomorphism $f : c_0 \to \ell_2$. *We define a second norm* $\|\cdot\|$ *on* c_0 *by*

$$\|x\| = \|f(x)\|_2.$$

Then $(c_0, \|\cdot\|)$ *is a Banach space. But* c_0 *and* ℓ_2 *are not linearly homeomorphic, since* ℓ_2 *is reflexive, and* c_0 *is not. Therefore the two norms on* c_0 *are incomparable. By Example 2.4 and Theorem 2.9,* $(c_0, \|\cdot\|)$ *is a Banach sequence space which is not a* BK *space.*

Theorem 2.11 *Let X, Y and Z be FH spaces with $X \subset Y \subset Z$. If X is closed in Z, then X is closed in Y.*

Proof. X is closed in $(Y, \mathcal{T}_Z|_Y)$, so in $(Y, \mathcal{T}_Y)$ by Theorem 2.9. $\square$

Theorem 2.12 *Let X and Y be FH spaces with $X \subset Y$, and E be a subset of X.*

Then

$$\mathrm{cl}_Y(E) = \mathrm{cl}_Y(\mathrm{cl}_X(E)), \text{ in particular } \mathrm{cl}_X(E) \subset \mathrm{cl}_Y(E).$$

Proof. Since $\mathcal{T}_Y|_X \subset \mathcal{T}_X$ by Theorem 2.9, it follows that $\mathrm{cl}_X(E) \subset \mathrm{cl}_Y(E)$. This implies

$$\mathrm{cl}_Y(\mathrm{cl}_X(E)) \subset \mathrm{cl}_Y(\mathrm{cl}_Y(E)) = \mathrm{cl}_Y(E).$$

Conversely, $E \subset \mathrm{cl}_X(E)$ implies $\mathrm{cl}_Y(E) \subset \mathrm{cl}_Y(\mathrm{cl}_X(E))$. $\square$

Example **2.13** *(a) Since c_0 and c are closed in ℓ_∞, their BK topologies are the same; since ℓ_1 is not closed in ℓ_∞, its BK topology is strictly stronger than that of ℓ_∞ on ℓ_1 (Theorem 2.9).*

(b) If c is not closed in an FK space X, then X must contain unbounded sequences (Theorem 2.11).

Definition 2.14 *Let $X \supset \phi$ be an FK space. Then X is said to have*
(a) **AD** *if cl$_X(\phi) = X$;*
(b) **AK** *if every sequence $x = (x_k)_{k=0}^\infty \in X$ has a unique representation*

$$x = \sum_{k=0}^\infty x_k e^{(k)} \text{ where } e_k^{(k)} = 1 \text{ and } e_j^{(k)} = 0 \ (j \neq k).$$

Example **2.15** *(a) Every FK space with AK has AD.*

(b) An Example of an FK space with AD which does not have AK can be found in [Wil2, Example 5.2.14, p. 80].

(c) The spaces ω, c_0 and ℓ_1 have AK.

(d) The space c does not have AK; every sequence $x = (x_k)_{k=0}^{\infty} \in c$ has a unique representation

$$x = \ell\, e + \sum_{k=0}^{\infty} (x_k - \ell) e^{(k)} \text{ where } \ell = \lim_{k \to \infty} x_k \text{ and } e_k = 1 \text{ for all } k.$$

(e) The space ℓ_∞ has no Schauder basis, since it is not separable.

Applications

Theorem 2.16 *Let X be an FK space with AD, and Y and Y_1 be FK spaces with Y_1 a closed subspace of Y.*
Then $A \in (X, Y_1)$ if and only if $A \in (X, Y)$ and $Ae^{(k)} \in Y_1$ for all k.

Proof. First we assume $A \in (X, Y_1)$.
$Y_1 \subset Y$ implies $A \in (X, Y)$, and $e^{(k)} \in X$ implies $Ae^{(k)} \in Y_1$.
Now we assume $A \in (X, Y)$ and $Ae^{(k)} \in Y_1$ for all k.
Define $f_A : X \to Y$ by $f_A(x) = Ax$ for all $x \in X$. First $Ae^{(k)} \in Y_1$ implies $f_A(\phi) \subset Y_1$. By Theorem 2.8, f_A is continuous, hence $f_A(\mathrm{cl}_X(\phi)) = \mathrm{cl}_Y(f_A(\phi))$. Since Y_1 is closed in Y, and ϕ is dense in X, we have

$$f_A(X) = f_A(\mathrm{cl}_X(\phi)) = \mathrm{cl}_Y(f_A(\phi)) \subset \mathrm{cl}_Y(Y_1) = \mathrm{cl}_{Y_1}(Y_1) = Y_1$$

by Theorem 2.9. $\square$

Theorem 2.17 *Let X be an FK space, $X_1 = X \oplus e$ and Y a linear subspace of ω.*
Then $A \in (X_1, Y)$ if and only if $A \in (X, Y)$ and $Ae \in Y$.

Proof. First we assume $A \in (X_1, Y)$.
$X \subset X_1$ implies $A \in (X, Y)$, and $e \in X_1$ implies $Ae \in Y$.

Conversely, we assume $A \in (X, Y)$ and $Ae \in Y$.
Let $x_1 \in X_1$ be given. Then there are $x \in X$ and $\lambda \in \mathbb{C}$ such that $x_1 = x + \lambda e$, and it follows that

$$Ax_1 = A(x + \lambda e) = Ax + \lambda Ae \in Y.$$

□

Let X and Y be Banach spaces. As usual, we denote by $\mathcal{B}(X, Y)$ set of all bounded linear operators $L : X \to Y$ which is a Banach space with the norm

$$\|L\| = \sup\left\{\|L(x)\| : \|x\| = 1\right\}.$$

Theorem 2.18 *Let X and Y be BK spaces.*

(a) Then $(X, Y) \subset \mathcal{B}(X, Y)$, that is every $A \in (X, Y)$ defines an operator $L_A \in \mathcal{B}(X, Y)$ by $L_A(x) = Ax$ for all $x \in X$.

(b) If X has AK then $\mathcal{B}(X, Y) \subset (X, Y)$.

(c) We have $A \in (X, \ell_\infty)$ if and only if

$$\|A\|_{(X,\ell_\infty)} = \sup_n \left(\sup\left\{|(Ax)_n| : \|x\| = 1\right\}\right) < \infty; \tag{2.2}$$

if $A \in (X, \ell_\infty)$ then

$$\|L_A\| = \|A\|_{(X,\ell_\infty)}. \tag{2.3}$$

Proof. (a) This is Theorem 2.8.

(b) Let $L \in \mathcal{B}(X,Y)$ be given.
We write $L_n = P_n \circ L$ for all n, and put $a_{nk} = L_n(e^{(k)})$ for all n and k.
Let $x = (x_k)_{k=0}^{\infty} \in X$ be given.
Since X has AK, $x = \sum_{k=0}^{\infty} x_k e^{(k)}$, and since Y is a BK space, $L_n \in X^*$ for all n. Hence

$$L_n(x) = \sum_{k=0}^{\infty} x_k L_n(e^{(k)}) = \sum_{k=0}^{\infty} a_{nk} x_k = (Ax)_n \text{ for all } n,$$

and so $L(x) = Ax$.

(c) The sufficiency of (2.2) is trivial.
Assume $A \in (X,Y)$.

Then $L_A \in \mathcal{B}(X, \ell_\infty)$ by Part (a), hence

$$\|L_A(x)\|_\infty \leq \|L_A\| \text{ for all } x \in X,$$

and (2.2) is an immediate consequence.

Also, (2.3) is obvious from the definitions of the operator norm and the norm $\|\cdot\|_{(X,\ell_\infty)}$. □

3. Multiplier and Dual Spaces

The so–called β–duals are of importance in the theory of matrix transformations. They are special cases of the multiplier spaces. Let cs and bs be the sets of all convergent and bounded series.

Definition 3.1 *Let X and Y be subsets of ω. Then*

$$M(X,Y) = \left\{a \in \omega : ax = (a_k x_k)_{k=0}^{\infty} \in Y \text{ for all } x \in X\right\}$$

is called the **multipler space of X in Y**. *Special cases are*

$$X^{\alpha} = M(X, \ell_1), \quad \text{the } \alpha- \textbf{ dual of X},$$
$$X^{\beta} = M(X, cs), \quad \text{the } \beta- \textbf{ dual of X},$$
$$X^{\gamma} = M(X, bs), \quad \text{the } \gamma- \textbf{ dual of X}.$$

Proposition 3.2 *Let* X, X_1, Y, $Y_1 \subset \omega$ *and* $\{X_\delta\}$ *be a collection of subsets of* ω. *Then*

(i) $Y \subset Y_1$ *implies* $M(X, Y) \subset M(X, Y_1)$

(ii) $X \subset X_1$ *implies* $M(X_1, Y) \subset M(X, Y)$

(iii) $X \subset M(M(X, Y), Y)$

(iv) $M(X, Y) = M\left(M(M(X, Y), Y), Y\right)$

(v) $M\left(\bigcup_\delta X_\delta, Y\right) = \bigcap_\delta M(X_\delta, Y)$.

Definition 3.3 *Let* $X \supset \phi$ *be an* FK *space and* X' *be the continuous dual of* X. *Then* $X^f = \{(f(e^{(n)}))_{n=0}^{\infty} : f \in X'\}$ *is called the* **functional dual of X**.

Theorem 3.4 *(a) Let $\dagger$ denote any of the symbols α, β and γ. Then*

$$X^{\alpha} \subset X^{\beta} \subset X^{\gamma} \subset X^{f} \text{ and } X \subset X^{\dagger\dagger}.$$

(b) Let $X \supset \phi$ be an FK space. Then

$$X^{f} = (cl_{X}(\phi))^{f}.$$

(c) If $X \subset Y$ then $X^{f} \supset Y^{f}$. If X is closed in Y then $X^{f} = Y^{f}$.

Example **3.5** *Let $X = c_0 \oplus z$ with z unbounded. Then X is a BK space, $X^{f} = \ell_1$ and $X^{ff} = \ell_{\infty}$, so $X \not\subset X^{ff}$.*

Theorem 3.6 *Let $X \supset \phi$ be an FK space.*

(a) If X has AK then $X^{\beta} = X^{f}$.

(b) If X has AD then $X^{\beta} = X^{\gamma}$.

Theorem 3.7 *Let $X \supset \phi$ be an FK space. Then $X^\beta \subset X'$; this means, that there is a linear one–to–one map $T : X^\beta \to X'$. If X has AK then T is onto.*

The map T of Theorem 3.7 is defined as follows

$$\begin{aligned} T :& X^\beta \to X' \\ & a \mapsto f_a \in X' \text{ where } f_a = \sum_{k=0}^{\infty} a_k x_k \text{ for all } x \in X. \end{aligned}$$

Theorem 3.8 *Let $X \supset \phi$ be an FK space. Then $X^f = X'$ if and only if X has AD.*

The following results do not extend to FK spaces, in general.

Theorem 3.9 *Let* $X \supset \phi$ *and* Y *be* BK *spaces. Then* $Z = M(X, Y)$ *is a* BK *space with*

$$\|z\| = \sup\{\|xz\| : \|x\| \leq 1\} \text{ for } z \in Z.$$

Theorem 3.10 *Let* $X \supset \phi$ *be a* BK *space. Then* X^f *is a* BK *space.*

Theorem 3.11 *Let* $X \supset \phi$ *be a* BK *space. Then*

$$X^{ff} \supset cl_X(\phi).$$

Hence, if X *has* AD*, then* $X \subset X^{ff}$*.*

Example **3.12** *(i)* $M(c_0, c) = \ell_\infty$; *(ii)* $M(c, c) = c$; *(iii)* $M(\ell_\infty, c) = c_0$.

Example **3.13** *Let* $\dagger$ *denote any of the symbols* α, β *or* γ. *Then*

$$\omega^\dagger = \phi;\ \phi^\dagger = \omega;\ c_0^\dagger = c^\dagger = \ell_\infty^\dagger = \ell_1$$
$$\ell_1^\dagger = \ell_\infty;\ \ell_p^\dagger = \ell_q\ (1 < p < \infty; q = p/(p-1)).$$

Example **3.14** *We have* $c^\beta = c^f = \ell_1$. *The map* T *of Theorem 3.7 is not onto. We consider* $\lim \in X'$. *If there were* $a \in X^f$ *with* $\lim a = \sum_{k=0}^\infty a_k x_k$ *then it would follow that* $a_k = \lim e^{(k)} = 0$, *hence* $\lim x = 0$ *for all* $x \in c$, *contradicting* $\lim e = 1$.

Example **3.15** *Let* $X = c_0 \oplus z$ *with* $z \in \ell_\infty$. *Then*

$$X^{ff} = \ell_1^f = \ell_\infty \supset X,$$

but X *does not have* AD, *hence the condition of Theorem 3.10 is not necessary.*

4. Matrix Transformations

We apply the results of the previous sections to give necessary and sufficient conditions on the entries of a matrix A to be in a class (X,Y).

The first two results concern the transpose A^T of a matrix A.

Theorem 4.1 *Let X be an FK space and Y be any set of sequences. If $A \in (X, Y)$ then $A^T \in (Y^\beta, X^f)$.*
If X and Y are BK spaces and Y^β has AD then

$$A^T \in (Y^\beta, \mathit{cl}_{X^f}(X^\beta)).$$

Theorem 4.2 *Let X and Z be BK spaces with AK and $Y = Z^\beta$. Then $(X, Y) = (X^{\beta\beta}, Y)$; furthermore*

$$A \in (X, Y) \textit{ if and only if } A^T \in (Z, X^\beta).$$

Remark 4.3 *The results of the previous sections yield the characterisations of the classes (X, Y) where X and Y are any of the spaces ℓ_p $(1 \leq p \leq \infty)$, c_0, c with the exceptions of (ℓ_p, ℓ_r) where both p,r $\neq$ 1,∞ (the characterisations are unknown), and of (ℓ_∞, c) (Schur's theorem 1.2) and (ℓ_∞, c_0) ([S–T,* **21 (21.1)**]*) (no functional analytic proof seems to be known).*
The class (ℓ_2, ℓ_2) was characterised by Crone ([Cro] or [Ruc, pp. 111–115]).

Example **4.4** *(a) We have $(c_0, \ell_\infty) = (c, \ell_\infty) = (\ell_\infty, \ell_\infty)$; furthermore $A \in (\ell_\infty, \ell_\infty)$ if and only if*

$$\|A\|_{(\infty,\infty)} = \sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty. \tag{4.1}$$

If A is in any of the classes above then

$$\|L_A\| = \|A\|_{(\infty,\infty)}.$$

(b) (Toeplitz's theorem 1.1) We have $A \in (c, c)$ *if and only if (4.1) holds and*

$$\lim_{n\to\infty} a_{nk} = \alpha_k \text{ exists for every } k; \tag{4.2}$$

and

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = \alpha \text{ exists.} \tag{4.3}$$

Proof. (a) We have $A \in (c_0, \ell_\infty)$ if and only if (4.1) by (2.2) in Theorem 2.18, and since $c_0^\beta = \ell_1$ and c_0^* and ℓ_1 are norm isomorphic.
Furthermore $c_0 \subset c \subset \ell_\infty$ implies $(\ell_\infty, \ell_\infty) \subset (c, \ell_\infty) \subset (c_0, \ell_\infty)$.
Also $(\ell_\infty, \ell_\infty) = (c_0^{\beta\beta}, \ell_\infty)$ by the first part of Theorem 4.2.
The last part is obvious from Theorem 2.18.
(b) This is an immediate consequence of Part (a), and Theorems 2.16 and

2.17. $\square$

Example **4.5** *We have* $(\ell_1, \ell_1) = \mathcal{B}(\ell_1, \ell_1)$ *and* $A \in (\ell_1, \ell_1)$ *if and only if*

$$\|A\|_{(1,1)} = \sup_k \sum_{n=0}^{\infty} |a_{nk}| < \infty. \tag{4.4}$$

If $A \in (\ell_1, \ell_1)$ *then*

$$\|L_A\| = \|A\|_{(\ell_1,\ell_1)}. \tag{4.5}$$

Proof. Since ℓ_1 has AK, Theorem 2.18 (b) yields the first part.

We apply the second part of Theorem 4.2 with $X = \ell_1$, $Z = c_0$, BK spaces with AK, and $Y = Z^\beta = \ell_1$ to obtain $A \in (\ell_1, \ell_1)$ if and only if $A^T \in (\ell_\infty, \ell_\infty)$; by Example 4.4 (a), this is the case if and only if (4.4) is satisfied.

Furthermore, if $A \in (\ell_1, \ell_1)$ then

$$\|L_A(x)\|_1 = \sum_{n=0}^{\infty} \left| \sum_{k=0}^{\infty} a_{nk} x_k \right| \le \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} |a_{nk} x_k| \le \|A\|_{(1,1)} \|x\|_1$$

implies

$$\|L_A\| \le \|A\|_{(1,1)}.$$

Also have $L_A \in \mathcal{B}(\ell_1, \ell_1)$ implies

$$\|L_A(x)\|_1 = \|Ax\|_1 \le \|L_A\| \, \|x\|_1,$$

and it follows from $\|e^{(k)}\|_1 = 1$ for all k that

$$\|A\|_{(1,1)} = \sup_k \sum_{n=0}^{\infty} |a_{nk}| = \sup_k \|L_A(e^{(k)})\|_1 \le \|L_A\|.$$

$\square$

5. Measures of Noncompactness

Now we find necessary and sufficient conditions for a matrix $A \in (X, Y)$ to define a compact operator L_A.

This can be achieved by applying the Hausdorff measure of noncompactness.

The first measure of noncompactness was defined and studied by Kuratowski ([Kur]), and later used by Darbo ([Dar]).

The Hausdorff measure of noncompactness was introduced and studied by Goldenstein, Gohberg and Markus ([GGM]).

Istrăţesku introduced and studied the Istrăţesku measure of noncompactness ([Ist]).

The interested reader is referred for measures on noncompactness to [AKP, B–G, Ist1, TBA, M–R].

We will only consider the Hausdorff measure of noncompactness; it is the most suitable one for our purposes.

Let (X, d) be a metric space, $x_0 \in X$ and $r > 0$. Then

$$B(x_0, r) = \{x \in X : d(x, x_0) < r\}$$

denotes the open ball of radius r, centred at x_0.

Definition 5.1 *Let (X, d) be a metric space and $\mathcal{M}$ denote the collection of bounded subsets of X. The function $\chi : \mathcal{M} \to [0, \infty)$ with*

$$\chi(Q) = \left\{ \varepsilon > 0 : Q \subset \bigcup_{k=0}^{n} B(x_k, r_k);\ x_k \in X,\ r_k < \varepsilon \right\}$$

is called **Hausdorff measure of noncompactness**; $\chi(Q)$ *is called the* **Hausdorff measure of noncompactness of** Q.

Proposition 5.2 *Let X be a metric space and Q,Q_1,$Q_2 \in \mathcal{M}$. Then*

$$\chi(Q) = 0 \text{ if and only if } Q \text{ is totally bounded,}$$
$$\chi(Q) = \chi(\bar{Q}),$$
$$Q_1 \subset Q_2 \text{ implies } \chi(Q_1) \leq \chi(Q_2),$$
$$\chi(Q_1 \cup Q_2) = \max\{\chi(Q_1), \chi(Q_2)\},$$
$$\chi(Q_1 \cap Q_2) \leq \min\{\chi(Q_1), \chi(Q_2)\}.$$

Proposition 5.3 *Let X be a normed space and Q,Q_1,$Q_2 \in \mathcal{M}$. Then*

$$\chi(Q_1 + Q_2) \leq \chi(Q1) + \chi(Q_2),$$
$$\chi(Q + x) = \chi(Q) \text{ for all } x \in X,$$
$$\chi(\lambda Q) = |\lambda|\chi(Q) \text{ for all scalars,}$$
$$\chi(Q) = \chi(\mathit{conv}(Q)).$$

Theorem 5.4 (Goldenstein, Gohberg,Markus) ([GGM])
Let X be a Banach space with a Schauder basis $(b_k)_{k=0}^{\infty}$, $Q \in \mathcal{M}$ and $\mathcal{P}_n : X \to X$ be the projector onto the linear span of $\{b_0, b_1, \ldots, b_n\}$. Then

$$(5.1)\quad \frac{1}{a} \limsup_{n\to\infty} \left(\sup_{x\in Q} \|(I - \mathcal{P}_n)(x)\| \right) \le \chi(Q) \le \le \limsup_{n\to\infty} \left(\sup_{x\in Q} \|(I - \mathcal{P}_n)(x)\| \right)$$

where

$$a = \limsup_{n\to\infty} \|I - \mathcal{P}_n\|.$$

So far we considered the measure of noncompactness of bounded subsets of a metric space. Now we define the measure of noncompactness of an operator.

Definition 5.5 *Let κ_1 and κ_2 be measures of noncompactness on the Banach spaces X and Y, and $\mathcal{M}_X$ and $\mathcal{M}_Y$ denote the collections of bounded sets in X and Y. An operator $L : X \to Y$ is said to be* $(\boldsymbol{\kappa_1}, \boldsymbol{\kappa_2})$–**bounded** *if*

$$L(Q) \in \mathcal{M}_Y \text{ for all } Q \in \mathcal{M}_X$$

and there exists a non–negative real c such that

$$\kappa_2(L(Q)) \leq c\,\kappa_1(Q) \text{ for all } Q \in \mathcal{M}_X.$$

If an operator L is (κ_1, κ_2)–bounded, then the number

$$\|L\|_{(\kappa_1,\kappa_2)} = \inf\{c \geq 0 : \kappa_2(L(Q)) \leq c\,\kappa_1(Q) \text{ for all } Q \in \mathcal{M}_X\}$$

is called the $(\boldsymbol{\kappa_1}, \boldsymbol{\kappa_2})$–**mesaure of noncompactness of L**.

If $\kappa = \kappa_1 = \kappa_2$, then we write $\|L\|_\kappa = \|L\|_{(\kappa,\kappa)}$.

Theorem 5.6 *Let X and Y be Banach spaces, $L \in \mathcal{B}(X,Y)$, S_X and $\bar{B}_X$ be the unit sphere and the closed unit ball in X, and χ be the Hausdorff measure of noncompactness.*
Then

$$\|L\|_\chi = \chi(L(S_X)) = \chi(L(\bar{B}_X)). \tag{5.2}$$

Theorem 5.7 *Let X, Y and Z be Banach spaces, $L \in \mathcal{B}(X,Y)$, $\tilde{L} \in \mathcal{B}(Y,Z)$, and $\mathcal{C}(X,Y)$ denote the set of compact operators in $\mathcal{B}(X,Y)$.*
Then $\|\cdot\|_\chi$ is a seminorm on $\mathcal{B}(X,Y)$ and

$$\begin{aligned} \|L\|_\chi = 0 \text{ if and only if } L \in \mathcal{C}(X,Y), \\ \|L\|_\chi \le \|L\|, \\ \|\tilde{L} \circ L\|_\chi \le \|\tilde{L}\|_\chi \, \|L\|_\chi. \end{aligned} \tag{5.3}$$

Theorem 5.8 *Let* $L \in \mathcal{B}(\ell_1, \ell_1)$, *and* A *denote the infinite matrix such that* $L(x) = Ax$ *for all* $x \in \ell_1$. *Then* $L \in \mathcal{C}(\ell_1, \ell_1)$ *if and only if*

$$\lim_{r\to\infty}\left(\sup_k \sum_{n=r}^{\infty} |a_{nk}|\right) = 0. \tag{5.4}$$

Proof. By Theorem 2.18 (b), every $L \in \mathcal{B}(X, Y)$ can be represented by a matrix $A \in (X, Y)$.

Writing $S = S_{\ell_1}$, we have by (5.2) in Theorem 5.6

$$\|L\|_\chi = \chi(L(S)).$$

For $r = 0, 1, \ldots$, let $A^{(r)}$ be the matrix with the first r rows replaced by 0. Then

$$\|(I - \mathcal{P}_{r-1})(L(x))\|_1 = \|A^{(r)}x\|_1$$

hence, by (4.4) in Example 4.5,

$$\sup_{x\in S}\|(I-\mathcal{P}_{r-1})(L(x))\|_1 = \|A^{(r)}\|_{(\ell_1,\ell_1)} = \sup_k \sum_{n=r}^{\infty}|a_{nk}|.$$

Since obviously $\|I-\mathcal{P}_{r-1}\| = 1$ for all r, and the limit on the right hand side of (5.4) exists, it follows from (5.1) in Theorem 5.4 that

$$\chi(L(S)) = \lim_{r\to\infty}\left(\sup_k \sum_{n=r}^{\infty}|a_{nk}|\right).$$

Finally it follows from (5.3) in Theorem 5.7 that $L \in \mathcal{C}(X,Y)$ if and only if (5.4) is satisfied. $\square$

Remark 5.9 *It follows from Theorem 5.8 and Example 4.5 that every* $L \in \mathcal{B}(X,Y)$ *is compact.*

References

[AKP] R. R. Akhmerov, M. I. Kamenskiĭ, A. S. Potapov, B. N. Sadovskiĭ, *Measures of Noncompactness and Condensing Operators*, Operator Theory: Advances and Applications, Vol. **55**, Birkhäuser, Basel, 1992 37

[B–G] J. Banaś, K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lecture Notes in Pure and Applied Mathematics, Vol. **60**, Marcel Dekker, New York, 1980 37

[Cro] L. Crone, A characterization of matrix mappings on ℓ^2, *Math. Z.* **123** (1971), 315–317 33

[Dar] G. Darbo, Punti uniti in transformazioni a condominio non compatto, *Rend. Sem. Math. Univ. Padova* **24** (1955), 84–92 37

[GGM] I. T. Gohberg, L. S. Goldenstein, A. S. Markus, Investigations of some properties with their q–norms, *Učen. Zap. Kishinevsk. Univ.* **29** (1957), 29–36 (Russian) 37, 40

[Har] G. H. Hardy, *Divergent Series*, Oxford University Press, 1973 4

[Ist] V. Istrăţesku, On a measure of noncompactness, *Bull. Math. Soc. Sci. Math. R. S. Roumanie (N.S.)* **16** (1972), 195–197 37

[Ist1] V. Istrăţesku, *Fixed Point Theory, An Introduction*, Reidel, Dordrecht, Boston and London, 1981 37

[K–G] P. K. Kamthan, M. Gupta, *Sequence Spaces and Series*, Marcel Dekker, New York, 1981 10

[Kur] K. Kuratowski, Sur les espaces complets, *Fund. Math* **15** (1930), 301–309 37

[Mad] I. J. Maddox, *Elements of Functional Analysis*, Cambridge University Press, 1971 4, 9

[Mad1] I. J. Maddox, *Infinite Matrices of Operators*, Lecture Notes in Mathematics **780**, Springer Verlag, Heidelberg–Berlin–New York, 1980 9

[M–R] E. Malkowsky, V. Rakočević, An introduction into the theory of sequence spaces and measures of noncompactness, *Zbornik radova, Matematički institut SANU* **9(17)** (2000), 143–234 9, 10, 37

[Pey] A. Peyerimhoff, *Lectures on Summability*, Lecture Notes in Mathematics **107**, Springer Verlag, Heidelberg–Berlin–New York, 1969 4

[Ruc] W. H. Ruckle, *Sequence Spaces*, Research Notes in Mathematics **49**, Pitman, Boston, London, Melbourne, 1981 33

[S–T] M. Stieglitz, H. Tietz, Matrixtransformationen in Folgenräumen, *Math. Z.* **154** (1977), 1–16 9, 33

[Toe] O. Toeplitz, Über allgemeine Mittelbildungen, *Prace. Mat. Fiz.* **22** (1911), 113–119 6

[TBA] J. M. Ayerbe Toledano, T. Dominguez Benavides, G. Lopez Acedo *Measures of Noncompactness in Metric Fixed Point Theory*, Operator Theory, Advances and Applications, Vol. **99**, Birkhäuser, Basel, Boston, Berlin, 1997 37

[Wil1] A. Wilansky, *Functional Analysis*, Blaisdell Publishing Company, New York, Toronto, London, 1964 10

[Wil2] A. Wilansky, *Summability through Functional Analysis*, North–Holland Mathematical Studies **85**, Elsevier Science Publishers, Amsterdam, New York, Oxford, 1984 9, 10, 20

[Zel] K. Zeller, Abschnittskonvergenz in FK–Räumen, *Math. Z.* **55** (1951), 55–70 10

[Z–B] K. Zeller, W. Beekmann, *Theorie der Limitierungsverfahren*, Springer Verlag, Heidelberg–Berlin–New York, 1968 4, 9