Pseudo–Spheres

A Sample of Electronic Lecture Notes in Mathematics

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0 The Description and Use of the Lecture Notes

The electronic lecture notes are typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$– LaTeX; it seems that LaTeX yields the best quality in display of mathematical formulae. We created the PDF file directly from the LaTeX source file by `\pdflatex` *filename*.

The graphics are included in PNG format in the LaTeX source code; the animations are in HTM format and can be linked to.

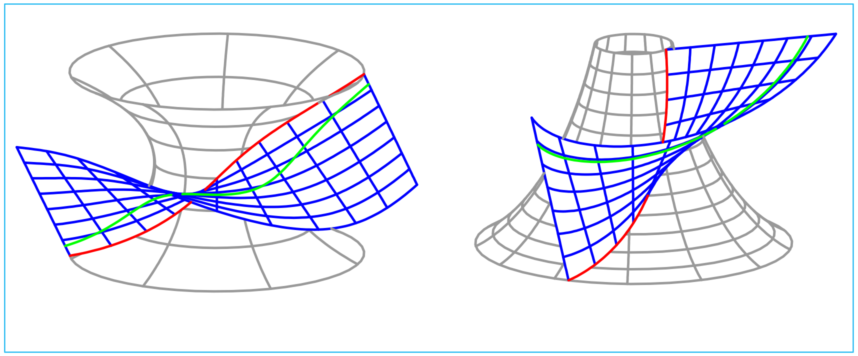
All our graphics and animations were created exclusively by the use of our own software package ([3, 1, 2, 4]) in Borland PASCAL, and also in DELPHI, which provides a user interface for interactive elements.

Our graphics created in PASCAL can be exported to various formats, including BMP, PS, PLT and JVX. These formats can then be converted by any graphics converter software to

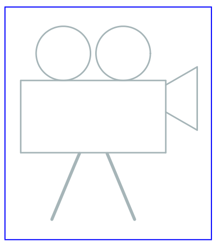
- an EPS file in TeX or LaTeX
- a PDF or PNG file
- a GIF file in HTML or WORD documents

We use the software package Animagic GIF 32 to create an animation in animated GIF format from a sequence of GIF files of our graphics, and include the animation as an animated GIF image in an HTML file.

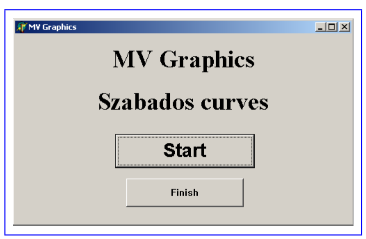
We decided to use small–sized PNG graphics in the lecture notes to have better control of the their placement in the text, and to reduce the time to open the lecture notes. The full–sized PDF graphics can be linked to by clicking on the PNG picture. To return to the lecture notes, we recommend to click on Close in the navigation bar at the top of the full–sized picture.



Our animations are linked to by clicking on the following icon



The interactive elements can be opened by clicking on



Finally we mention that the best quality of viewing is obtained in Full Screen mode.

1 A Study of Pseudo–Spheres

Here we study the most important geometrical properties of *pseudo–spheres*.

1.1 Intoduction

The study of pseudo–spheres is part of every course on Differential Geometry. We introduce pseudo–spheres as surfaces of rotation generated by tractrices (Definition 1.1). A parametric representation a tractrix is derived in Proposition 1.2 from its geometric definition in Definition 1.1. In Proposition 1.3, we establish a parametric representation for pseudo–spheres.

We also give a characterisation of pseudo–spheres as surfaces of rotation with constant negative *Gaussian curvature* (Remark 1.4), and of the related hyperbolic, elliptic spherical, and parabolic, hyperbolic and elliptic pseudo–spherical surfaces.

Furthermore, we solve the differential equation for *asymptotic lines* on pseudo–spheres (Proposition 1.5), and give a parametric representation for the asymptotic lines with respect to their arc lenghts (Proposition 1.6). This parametric representation is used to compute the *vectors of the trihedra*, the *curvature* and *torsion* along the asymptotic line (Proposition 1.8), and to determine the *circles* (Proposition 1.10 (a)) and *spheres* (Proposition 1.10 (b)).

Finally, we solve the differential equations for the *geodesic lines* on pseudo–spheres (Proposition 1.12).

1.2 The Tractrix and Pseudo–Spheres

Pseudo–spheres are generated by the rotation of so-called *tractrices* in the x^1x^3–plane about the x^3–axis.

Definition 1.1 The tractrix
Let γ and γ^* be a curve and a straight line in a plane, respectively, that have no points of intersection. Given a point P on γ, letP^* denote the point of intersection of γ^* with the tangent to γ at P. If the distances between the points P and P^* have a constant value d, the curve γ is called a **tractrix** *(Figure 1).*

Figure 1 *The construction of a tractrix*

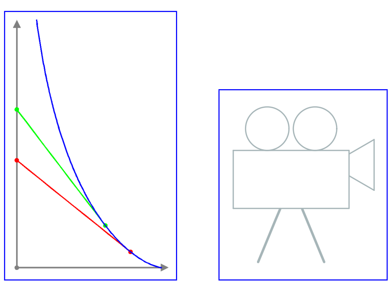
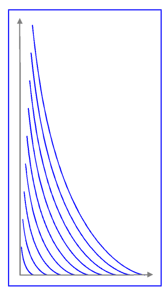


Figure 2 *A familiy of tractrices*



We derive a parametric representation for the tractrix.

Proposition 1.2 *We introduce a Cartesian coordinatesystem in the plane such that the straight line γ^* is the y–axis. Then γ has a parametric representation*

$$\vec{x}(\phi) = \left\{ d\sin\phi, d\left(\log\left(\tan\left(\frac{\phi}{2}\right)\right) + \cos\phi\right)\right\} \quad (\phi \in (0, \pi/2)). \tag{1.1}$$

Proof. Let $\vec{x}(t) = \{t, y(t)\}$ be a parametric representation of γ. Then the tangent to γ at t is given by $\vec{x}(t) + \lambda\{1, y'(t)\}$ ($\lambda \in \mathbb{R}$), and so we obtain for P^*

$$\overrightarrow{OP^*} = \{0, p^*\} = \{t, y(t)\} + \lambda\{1, y'(t)\}, \text{ that is } \lambda = -t.$$

We observe that $\gamma \cap \gamma^* = \emptyset$ implies $d > 0$, hence we have

$$d = d(P, P^*) = \|\vec{x}(t) - (\vec{x}(t) - t\{1, y'(t)\})\| = |t|\sqrt{1 + (y'(t))^2}$$

and so, since $|t| > 0$,

$$y'(t) = \pm\frac{\sqrt{d^2 - t^2}}{|t|} \text{ for } |t| < d. \tag{1.2}$$

We choose the upper sign in (1.2) and $t \in (0, d)$, and obtain

$$y(t) = \int \frac{\sqrt{d^2 - t^2}}{t}\, dt. \tag{1.3}$$

Since $t \in (0, d)$, we may put $t = d\sin\phi$ for $\phi \in (0, \pi/2)$ and this yields

$$\begin{aligned}\int \frac{\sqrt{d^2 - t^2}}{t}\, dt &= \int \frac{\sqrt{d^2 - d^2\sin^2\phi}}{d\sin\phi} d\cos\phi\, d\phi \\ &= d\int \frac{\cos^2\phi}{\sin\phi}\, d\phi = d\,(I_1 + I_2)\,,\end{aligned} \tag{1.4}$$

where

$$I_1 = \int \frac{d\phi}{\sin\phi}$$

and

$$I_2 = \int \sin\phi\, d\phi = -\cos\phi = \sqrt{1 - \sin^2\phi} = \frac{\sqrt{d^2 - t^2}}{d} + c_2 \tag{1.5}$$

where c_2 is a constant of integration. To evaluate the integral I_1, we make the substitution $z = \tan(\phi/2)$. Then $z \in (0,1)$ and we obtain

$$\begin{aligned}\frac{d\phi}{dz} &= \frac{d}{dz}(2\arctan z) = \frac{2}{1+z^2},\\ \sin\phi &= 2\sin(\phi/2)\cos(2\phi/2) = 2\tan(\phi/2)\cos^2(\phi/2)\\ &= 2\tan(\phi/2)\frac{\cos^2(\phi/2)}{\cos^2(\phi/2)+\sin^2(\phi/2)}\\ &= \frac{2\tan(\phi/2)}{1+\tan^2(\phi/2)} = \frac{2x}{1+x^2}\end{aligned}$$

and

$$\begin{aligned}I_1 &= \int \frac{1+z^2}{2z}\,\frac{2}{1+z^2}\,dz = \int \frac{dz}{z} = \log z + c_1\\ &= \log\left(\tan\left(\frac{\phi}{2}\right)\right) + c_1, \qquad (1.6)\end{aligned}$$

where c_1 is a constant of integration. We put $c = d(c_1+c_2)$. Then it follows from (1.3), (1.4), (1.5) and (1.6) that

$$y(\phi) = d\left(\log\left(\tan\left(\frac{\phi}{2}\right)\right) + \cos\phi\right) + c.$$

If we choose c such that $\lim_{\phi\to\pi/2} y(\phi) = 0$, then we have $c = 0$, and (1.1) is an immediate consequence. □

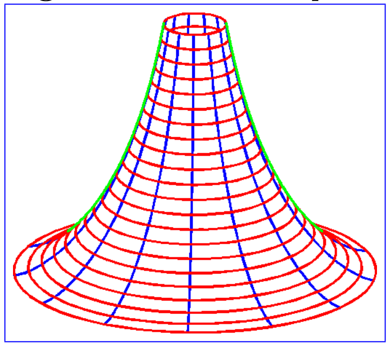
We recall that a surface of rotation is generated by the rotation of a curve γ in the x^1x^3–plane about the x^3-axis. If γ is given by a parametric representation $\vec{x}(t) = \{r(t), 0, h(t)\}$ for t in some open interval I, where r and h are continuously differentiable on I with $r(t) > 0$ on I, then we write $u^1 = t$ and u^2 for the angle of rotation, and obtain the following parametric representation for the surface of rotation $RS(\gamma)$ generated by the curve γ

$$\vec{x}(u^i) = \{r(u^1)\cos u^2, r(u^1)\sin u^2, h(u^1)\}$$
$$((u^1,u^2) \in I \times (0,2\pi)). \quad (1.7)$$

Proposition 1.3 *The pseudo–sphere generated by the tractrix given by (1.1) has a parametric representation*

$$\vec{x}(u^i) = \left\{e^{-u^1}\cos u^2, e^{-u^1}\sin u^2, \int \sqrt{d^2 - e^{-2u^1}}\,du^1\right\}$$
$$((u^1,u^2) \in (\log(1/d), \infty) \times (0,2\pi)) \text{ (Figure 3)}. \quad (1.8)$$

Figure 3 *A Pseudo–sphere*



Proof. Writing $t = t(t^*) = e^{-t^*} > 0$ and $\vec{y}^* = \vec{y}(t(t^*))$, we obtain from (1.2) with the lower sign

$$\begin{aligned}\frac{d\vec{y}^*}{dt^*} &= \frac{d\vec{y}}{dt}\,\frac{dt}{dt^*} = -\frac{\sqrt{d^2 - e^{-2t^*}}}{e^{-t^*}}\left(-e^{-t^*}\right)\\ &= \sqrt{d^2 - e^{-2t^*}} \quad \text{for } t^* > \log(1/d).\end{aligned}$$

We put $u^1 = t^*$ and the parametric representation (1.8) is an immediate consequence of (1.7). $\square$

Remark 1.4 *Pseudo–spheres are surfaces of revolution of constant negative* Gaussian curvature *K.*
*If we assume that $K : I \to I\!R$ is a given function and write $u = u^1$, for short, then the surface of rotation that has K as its Gaussian curvature is given by (*for details*)*

$$r''(u) + K(u)r(u) = 0 \quad \text{and} \quad h(u) = \pm \int \sqrt{1-(r'(u))^2}\,du, \tag{1.9}$$

and we may choose the upper sign for h without loss of generality.
Surfaces of rotation with a given constant Gaussian curvature are called spherical *or* pseudo–spherical surfaces *depending on whether $K > 0$ or $K < 0$.*
First we assume $K > 0$ and put $K = 1/c^2$ for some constant $c > 0$. Then the general solution of the differential equation in (1.9) is

$$r(u) = \lambda \cos\left(\frac{u}{c}\right) \text{ with } \lambda > 0$$

and we obtain

$$h(u) = \int \sqrt{1 - \frac{\lambda^2}{c^2}\sin^2\left(\frac{u}{c}\right)}\,du. \tag{1.10}$$

There three different types of spherical surfaces corresponding to the cases $\lambda = c$, $\lambda > c$ or $\lambda < c$.

Case 1. $\lambda = c$

Then the surface has a parametric representation

$$\vec{x}(u^i) = \left(c\cos\left(\frac{u^1}{c}\right)\cos u^2, c\cos\left(\frac{u^1}{c}\right)\sin u^2, c\sin\left(\frac{u^1}{c}\right)\right)$$
$$((u^1,u^2) \in (-\pi/2, \pi/2) \times (0, 2\pi)).$$

This is a sphere with radius c and centre in the origin.

Case 2. $\lambda > c$

The corresponding surfaces are called hyperbolic spherical surfaces*. Now the integral for h in (1.10) only exists for values of u with*

$$\left|\sin\left(\frac{u}{c}\right)\right| \le \frac{c}{\lambda},$$

that is

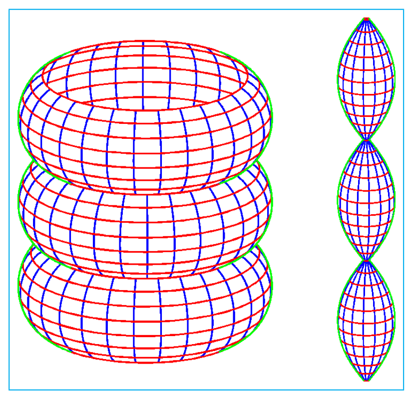
$$u \in I_k = \left[-c\arcsin\left(\frac{c}{\lambda}\right) + k\pi, c\arcsin\left(\frac{c}{\lambda}\right) + k\pi\right]$$

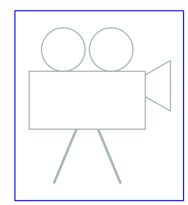
for $k = 0, \pm 1, \pm 2, \ldots$ (left in Figure 4). Every interval I_k defines a region of the surface. The radii of the circles of the u^2–lines are minimal at the end points of the intervals I_k and equal to $r = \sqrt{\lambda^2 - c^2}$, whereas the maximum radius $R = \lambda$ is attained in the middle of each region (left in Figure 6).

Case 3. $\lambda < c$

The corresponding surfaces are called elliptic spherical surfaces *(right in Figure 4). Now the integral for h in (1.10) exists for all u and the radii r of the circles of the u^2–lines attain all values $r \le \lambda$.*

Figure 4 *Spherical surfaces*





Now we assume $K < 0$ and put $K = -1/c^2$ for some constant $c > 0$. The general solution of the differential equation in (1.9) is

$$r(u) = C_1 \cosh\left(\frac{u}{c}\right) + C_2 \sinh\left(\frac{u}{c}\right) \quad \text{with constants } C_1 \text{ and } C_2.$$

Case 1. $C_1 = -C_2 = \lambda \neq 0$

The corresponding surfaces are called parabolic pseudo–spherical surfaces *(left in Figure 5). They have a parametric representation with*

$$r(u^1) = \lambda \exp\left(-\frac{u^1}{c}\right) \text{ and}$$

$$h(u^1) = \int \sqrt{1 - \frac{\lambda^2}{c^2}\exp\left(-\frac{2u^1}{c}\right)}\, du^1 \text{ for } u^1 > c\log(|\lambda|/c).$$

Case 2. $C_2 = 0$ and $C_1 = \lambda \neq 0$

The corresponding surfaces are called hyperbolic pseudo–spherical surfaces *(middle in Figure 5). They have a parametric representation with*

$$r(u^1) = \lambda \cosh\left(\frac{u^1}{c}\right) \text{ and } h(u^1) = \int \sqrt{1 - \frac{\lambda^2}{c^2}\sinh^2\left(\frac{u^1}{c^2}\right)}\, du^1$$

$$\text{for } |u^1| \leq c \cdot \mathit{arsinh}\left(\frac{c}{|\lambda|}\right) = c\log\left(\frac{c}{|\lambda|} + \sqrt{\frac{c^2}{\lambda^2}+1}\right).$$

The radii r of the circles of the u^2–lines satisfy $|\lambda| \leq r \leq \sqrt{\lambda^2 + c^2}$ (right in Figure 6).

Case 3. $C_1 = 0$ and $C_2 = \lambda \neq 0$

The corresponding surfaces are called elliptic pseudo–spherical surfaces. *They have a parametric representation with*

$$r(u^1) = \lambda \sinh\left(\frac{u^1}{c}\right)$$

and

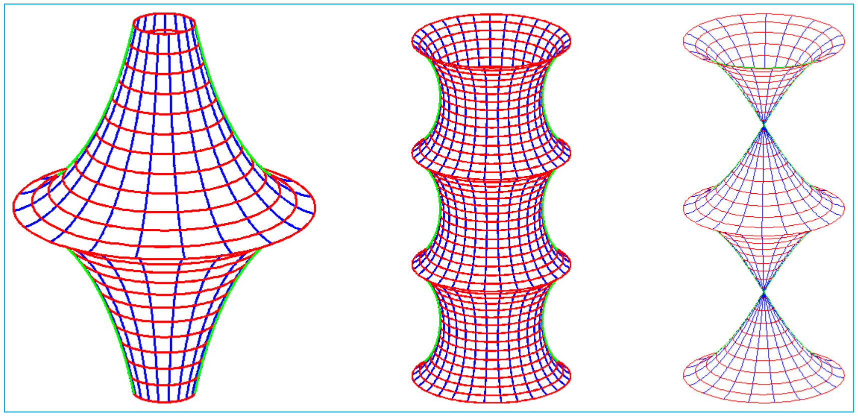
$$h(u^1) = \int \sqrt{1 - \frac{\lambda^2}{c^2}\cosh^2\left(\frac{u^1}{c}\right)}\, du^1 \tag{1.11}$$

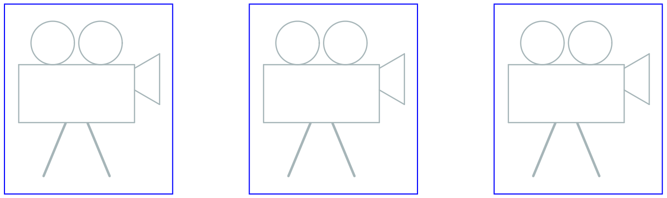
for all u^1 with

$$\cosh\left(\frac{u^1}{c}\right) \leq \frac{c}{|\lambda|};$$

(since $\cosh u^1 \geq 1$ for all u^1, we must have $|\lambda| \leq c$) (right in Figure 5); the integral for h in (1.11) is elliptic. The radii r of the circles of the u^2–lines satisfy $0 \leq r \leq \sqrt{c^2 - \lambda^2}$.

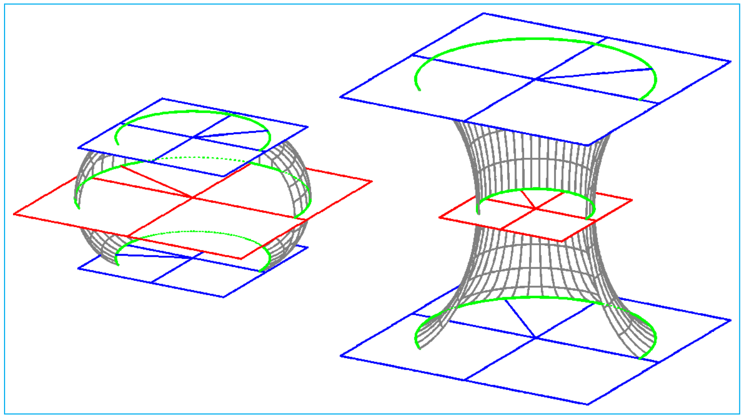
Figure 5 *Pseudo–spherical surfaces*





parabolic *hyperbolic* *elliptic*

Figure 6 *Hyperbolic spherical and pseudo–spherical surfaces with minimal and maximal radii of u^2–lines*



1.3 Aymptotic Lines on a Pseudo–Sphere

We recall that the second fundamental coefficients L_{ik} $(i,k=1,2)$ of a surface of rotation with a parametric representation

$$\vec{x}(u^i) = \{r(u^1)\cos u^2, r(u^1)\sin u^2, h(u^1)\}$$

where $r(u^1) > 0$ and $|r'(u^1)| + |h'(u^1| > 0$ are given by

$$L_{11}(u^i) = L_{11}(u^1) = \frac{h''(u^1)r'(u^1) - h'(u^1)r''(u^1)}{\sqrt{(r'(u^1))^2 + (h'(u^1))^2}},$$
$$L_{12}(u^i) = L_{21}(u^i) = 0$$

and

$$L_{22}(u^i) = L_{22}(u^1) = \frac{h'(u^1)r(u^1)}{\sqrt{(r'(u^1))^2 + (h'(u^1))^2}}.$$

In the case of the pseudo–sphere, we have $(r')^2 + (h')^2 = 1$, hence $r''r' + h''h' = 0$ and so $h'' = -r''r'/h'$, and the second fundamental coefficients are given by

$$\begin{aligned} L_{11} = h''r' - h'r'' &= -\frac{(r')^2 r''}{h'} - r''h' \\ &= -\frac{r''}{h'}\left((r')^2 + (h')^2\right) = -\frac{r''}{h'} \end{aligned} \tag{1.12}$$

and

$$L_{22} = h'r. \tag{1.13}$$

Aysmptotic lines on a surface with second fundamental coefficients L_{ik} $(k = 1,2)$ are the real solutions of the differential equations

$$L_{11}(u^i)\left(du^1\right)^2 + 2L_{12}(u^i)du^1du^2 + L_{22}(u^i)\left(du^2\right)^2 = 0; \tag{1.14}$$

they only exist in neighbourhoods of so–called *hyperbolic points* of the surface, that is points with $L = L_{11}L_{22} - (L_{12})^2 > 0$.

In the case of surfaces of rotation, the differential equation (1.14) for asymptotic lines reduces to

$$\frac{du^2}{du^1} = \pm\sqrt{-\frac{L_{11}(u^1)}{L_{22}(u^1)}} = \sqrt{\frac{r''(u^1)}{r(u^1)(h'(u^1))^2}} \quad \text{for } r(u^1)r''(u^1) > 0. \qquad (1.15)$$

Now we give the asymptotic lines on the pseudo–sphere.

Proposition 1.5 *The asymptotic lines on the pseudo–sphere with a parametric representation*

$$\vec{x}(u^i) = \left\{e^{-u^1}\cos u^2, e^{-u^1}\sin u^2, \int\sqrt{1-e^{-2u^1}}\,du^1\right\} \quad \text{for } (u^1,u^2) \in (0,\infty)\times(0,2\pi) \qquad (1.16)$$

((1.8) with $d = 1$) are given by

$$u_1(t) = t \text{ and } u_2(t) = \pm\log\left(\frac{1+\sqrt{1-e^{-2t}}}{e^{-t}}\right) + c \quad \text{for all } t > 0, \qquad (1.17)$$

where $c \in I\!R$ is a constant of intgration (Figure 7).

Proof. Again, we write $u = u^1$, and observe that

$$r(u) = \mathrm{e}^{-u},\ h'(u) = \sqrt{1-\mathrm{e}^{-2u}} \text{ and } r''(u) = r(u).$$

Hence the differential equation (1.15) for the asymptotic lines reduces to

$$\frac{du^2}{du^1} = \pm\frac{1}{|h'(u)|} = \pm\frac{1}{h'(u)},$$

since $h'(u) > 0$ for all $u > 0$. Therefore we have to solve the integral

$$I = \int\frac{du}{\sqrt{1-\mathrm{e}^{-2u}}}.$$

The substitution $t = \mathrm{e}^{-u}$ yields

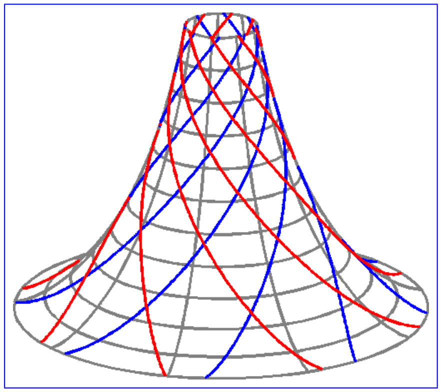
$$I = -\int\frac{dt}{t\sqrt{1-t^2}} = \int\frac{1}{1-t^2}\left(-\frac{dt}{t^2}\right).$$

Now we put $z = 1/t$, and obtain $dz = -dt/t^2$ and

$$I = \int\frac{dz}{z\sqrt{1-\frac{1}{z^2}}} = \int\frac{dz}{z^2-1} = \log\left(z+\sqrt{z^2-1}\right)$$
$$= \log\left(\frac{1+\sqrt{1-t^2}}{t}\right) = \log\left(\frac{1+\sqrt{1-\mathrm{e}^{-2u}}}{\mathrm{e}^{-u}}\right).$$

□

Figure 7 *Families of asymptotic lines on a speudo–sphere*



We choose the upper sign and $c = 0$ in (1.17), that is, we consider the aysmptotic line given by

$$u_1(t) = t \text{ and } u_2(t) = \log\left(\frac{1+\sqrt{1-e^{-2t}}}{e^{-t}}\right) \text{ for all } t > 0, \tag{1.18}$$

It is useful to obtain a parametric representation for the asymptotic line with respect to its arc length s.

Proposition 1.6 *The asymptotic line given by (1.18) has a parametric representation*

$$\vec{x}^*(s) = \left\{\frac{1}{\cosh s}\cos s, \frac{1}{\cosh s}\sin s, s - \tanh s\right\} \quad \textit{for all } s > 0. \tag{1.19}$$

Proof. We recall that the first fundamental coefficients for surfaces of rotation are given by

$$\begin{aligned} g_{11}(u^i) &= g_{11}(u^1) = (r'(u^1)^2 + (h'(u^1))^2, \\ g_{12}(u^i) &= g_{21}(u^i) = 0 \end{aligned}$$

and

$$g_{22}(u^i) = g_{22}(u^1) = (r(u^1))^2.$$

Therefore, we obtain for the pseudo–sphere with a parametric representation (1.16)

$$g_{11}(u^1) = 1 \text{ and } g_{22}(u^1) = e^{-2u^1},$$

hence for the aymptotic line given by (1.18)

$$\begin{aligned} \|\vec{x}'(t)\|^2 &= g_{11}(u^1(t))\left(\frac{du^1(t)}{dt}\right)^2 + g_{22}(u^1(t))\left(\frac{du^2(t)}{dt}\right)^2 \\ &= 1 + \frac{e^{-2t}}{1-e^{-2t}} = \frac{1}{1-e^{-2t}}. \end{aligned}$$

Now it follows for the arc length of the asymptotic line by (1.17)

$$\begin{aligned} s(t) &= \int \|\vec{x}'(t)\|\, dt = \int \frac{dt}{\sqrt{1-e^{-2t}}} \\ &= \log\left(\frac{1+\sqrt{1-e^{-2t}}}{t}\right) = u^2(t). \end{aligned}$$

This implies

$$e^s e^{-t} - 1 = \sqrt{1-e^{-2t}} \text{ and } e^{-2t}\left(e^{2s}+1\right) - 2e^{-t}e^s = 0,$$

that is, since $e^{-t} > 0$

$$e^{-t} = \frac{2e^s}{e^{2s}+1} = \frac{1}{\cosh s}. \tag{1.20}$$

Furthermore, we have

$$\begin{aligned} h(s) = h(t(s)) &= \int \sqrt{1-e^{-2t(s)}}\frac{dt(s)}{ds}\, ds \\ &= \int \sqrt{1-\frac{1}{\cosh^2(s)}}\sqrt{1-\frac{1}{\cosh^2(s)}}\, ds \\ &= \int \left(1-\frac{1}{\cosh^2(s)}\right) ds = s - \tanh(s). \end{aligned}$$

This and (1.20) together yield the parametric representation (1.19). □

Remark 1.7 *Another, simpler proof is to check if*

$$\|\dot{\vec{x}}^*(s)\| = 1 \text{ where } \dot{\vec{x}}^*(s) = \frac{d\vec{x}^*(s)}{ds}.$$

Writing $\phi(s) = 1/\cosh s$, *we obtain*

$$\dot{\vec{x}}^*(s) = \left\{\dot\phi(s)\cos s, \dot\phi(s)\sin s, 1-\phi^2(s)\right\} + \{-\phi(s)\sin s, \phi(s)\cos s, 0\}$$

and

$$\begin{aligned}\|\dot{\vec{x}}^*(s)\|^2 &= (\dot\phi(s))^2 + (1-\phi^2(s))^2 + \phi^2(s)\\ &= (\dot\phi(s))^2 + 1 - \phi^2(s) + \phi^4(s).\end{aligned}$$

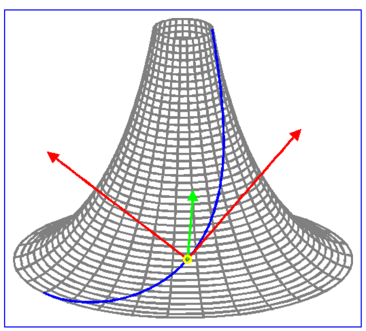
Since

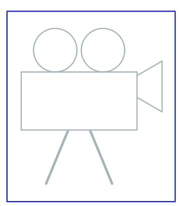
$$\begin{aligned}(\dot\phi(s))^2 &= \left(\frac{d}{ds}\left(\frac{1}{\cosh s}\right)\right)^2 = \frac{\sinh^2(s)}{\cosh^4(s)}\\ &= \frac{\cosh^2(s)-1}{\cosh^4(s)} = \phi^2(s) - \phi^4(s),\end{aligned}$$

we obtain

$$\|\dot{\vec{x}}^*(s)\|^2 = \phi^2(s) - \phi^4(s) + 1 - \phi^2(s) + \phi^4(s) = 1 \text{ for all } s > 0.$$

Figure 8 *The vectors of a trihedron*





1.4 The Vectors of the Trihedra and the Curvature along an Asymptotic Line

Now we compute the vectors of the trihedra and the curvature along the aymptotic line given by (1.18).

Proposition 1.8 *The vectors $\vec{v}_k(s)$ $(k =, 1, 2, 3)$ of the trihedra (Figure 8), the curvature $\kappa(s)$ and torsion $\tau(s)$ along the asymptotic line given by (1.19) are*

$$\begin{aligned}\vec{v}_1(s) &= -\frac{\sinh s}{\cosh^2(s)}\{\cos s, \sin s, -\sinh s\}\\ &\quad + \frac{1}{\cosh s}\{-\sin s, \cos s, 0\}, &(1.21)\\ \vec{v}_2(s) &= -\frac{1}{\cosh^2(s)}\{\cos s, \sin s, -\sinh s\}\\ &\quad - \frac{\sinh s}{\cosh s}\{-\sin s, \cos s, 0\}, &(1.22)\\ \vec{v}_3(s) &= \frac{1}{\cosh s}\{\sinh s\cos s, \sinh s\sin s, 1\} &(1.23)\\ \kappa(s) &= \frac{2}{\cosh s}. &(1.24)\end{aligned}$$

and

$$\tau(s) = 1. \qquad (1.25)$$

Proof. We omit the arguement s in the function ϕ. We already know from Remark 1.7 that the tangent vector is

$$\begin{aligned}\vec{v}_1(s) = \dot{\vec{x}}^*(s) &= \{\dot\phi\cos s, \dot\phi\sin s, 1-\phi^2\} + \{-\phi\sin s, \phi\cos s, 0\}\\ &= -\frac{\sinh s}{\cosh^2(s)}\{\cos s, \sin s, -\sinh s\} + \frac{1}{\cosh s}\{-\sin s, \cos s, 0\}\end{aligned}$$

which is (1.21).
This yields

$$\ddot{\vec{x}}^*(s) = \{(\ddot{\phi} - \phi)\cos s, (\ddot{\phi} - \phi)\sin s, -2\dot{\phi}\phi\} + 2\{-\dot{\phi}\sin s, -\dot{\phi}\cos s, 0\}.$$

It follows from

$$\dot{\phi} = -\frac{\sinh s}{\cosh^2(s)} = -\phi^2 \sinh s$$

that

$$\begin{aligned}\ddot{\phi} &= -\phi^2\cosh s - 2\dot{\phi}\phi\sinh s = -\phi + 2\phi^3\sinh^2 s \\ &= -\phi + 2\phi - 2\phi^3 = \phi - 2\phi^3\end{aligned}$$

and

$$\ddot{\phi} - \phi = -2\phi^3.$$

Therefore, we have

$$\ddot{\vec{x}}^*(s) = -\frac{2}{\cosh^3(s)}\{\cos s, \sin s, -\sinh s\} - 2\frac{\sinh s}{\cosh^2(s)}\{-\sin s, \cos s, 0\}.$$

Now we obtain for the curvature $\kappa(s)$ along the asymptotic line

$$\begin{aligned}\kappa^2(s) = \|\ddot{\vec{x}}^*(s)\|^2 &= 4\left(\frac{1+\sinh^2(s)}{\cosh^6(s)} + \frac{\sinh^2(s)}{\cosh^4(s)}\right) \\ &= 4\frac{1}{\cosh^4(s)}(1+\sinh^2(s)) = \frac{4}{\cosh^2(s)},\end{aligned}$$

hence

$$\kappa(s) = \frac{2}{\cosh s}$$

which is (1.24).
Now the principal normal vectors $\vec{v}_2(s)$ along the asymptotic line are given by

$$\begin{aligned}\vec{v}_2(s) &= \frac{1}{\kappa(s)}\ddot{\vec{x}}^*(s) \\ &= -\frac{1}{\cosh^2(s)}\{\cos s, \sin s, -\sinh s\} - \frac{\sinh s}{\cosh s}\{-\sin s, \cos s, 0\}\end{aligned}$$

which is (1.22).
Furthermore, since

$$\begin{aligned}\vec{b}(s) =& \{\cos s, \sin s, -\sinh s\} \times \{-\sin s, \cos s, 0\} \\ &= \{\sinh s\cos s, \sinh s\sin s, 1\},\end{aligned}$$

the binormal vectors $\vec{v}_3(s)$ along the asymptotic line are given by

$$\begin{aligned}\vec{v}_3(s) = \vec{v}_1(s) \times \vec{v}_2(s) &= \left(\frac{\sinh^2(s)}{\cosh^3(s)} + \frac{1}{\cosh^3(s)}\right)\vec{b} \\ &= \frac{1}{\cosh s}\{\sinh s\cos s, \sinh s\sin s, 1\}\end{aligned}$$

which is (1.23).
Finally, we obtain from (1.23)

$$\dot{\vec{v}}_3(s) = \frac{1}{\cosh^2 s}\{\cos s, \sin s, -\sinh s\} + \frac{\sinh s}{\cosh s}\{-\sin s, \cos s, 0\} \tag{1.26}$$

and so by (1.23)

$$\begin{aligned}\tau(s) = -\vec{v}_2(s) \bullet \dot{\vec{v}}_3(s) &= \frac{1}{\cosh^4(s)}(1+\sinh^2(s)) + \frac{\sinh^2 s}{\cosh^2 s} \\ &= \frac{1}{\cosh^2 s}(1+\sinh^2 s) = 1.\end{aligned}$$

□

Remark 1.9 *(a) We can easily check the result (1.23) for the binormal vector of the asymptotic line by using the well–known fact, that the osculating planes and the tangent planes of a surface along an asymptotic line with non–vanishing curvature coincide, that means that*

$$\vec{v}_3(s) = \pm\vec{N}(u^i(s))$$

where

$$\vec{N}(u^i) = \frac{\vec{x}_1(u^i)\times\vec{x}_2(u^i)}{\|\vec{x}_1(u^i)\times\vec{x}_2(u^i)\|}$$

are the surface normal vectors of the surface with a parametric representation $\vec{x}(u^i)$. We note that the surface normal vectors of a surface of rotation are

$$\vec{N}(u^i) = \frac{1}{(r'(u^1))^2+(h'(u^1))^2}\{-h'(u^1)\cos u^2, -h(u^1)\sin u^2, r'(u^1)\}.$$

We already know that $(r'(u^1))^2+(h'(u^1))^2 = 1$ for the pseudo–sphere, and for the asymptotic line

$$h'(u^1(s)) = \sqrt{1-\frac{1}{\cosh^2 s}} = \frac{|\sinh s|}{\cosh s} = \frac{\sinh s}{\cosh s},$$

since $s > 0$,

$$r'(u^1(s)) = -r(u^1(s)) = \frac{1}{\cosh s}$$

and $u^2(s) = s$. Thus the surface normal vectors of the pseudo–sphere along the asymptotic line are given by

$$\vec{N}(u^i(s)) = \frac{1}{\cosh s}\{-\sinh s\cos s, -\sinh s\sin s, 1\} = -\vec{v}_3(s).$$

(b) The pseudo–sphere has constant Gaussian curvature $K(u^i) = -1$. Therefore we have by the Beltrami-Enneper theorem for the torsion along the asymptotic line

$$|\tau(s)| = \sqrt{-K(u^i(s))} = 1.$$

1.5 The Osculating Circles and Spheres along an Asymptotic Line on a Pseudo–Sphere

Now we consider the osculating circles and spheres along the asymptotic line on our pseudo–sphere.

Proposition 1.10 *Let γ be the asymptotic line with a parametric representation (1.19).*
(a) Then the centres of curvature along γ are given by

$$\vec{x}_m(s) = \left\{\frac{\cos s}{2\cosh s}, \frac{\sin s}{2\cosh s}, s-\frac{1}{2}\tanh s\right\} - \frac{1}{2}\sinh s\{-\sin s, \cos s, 0\}, \quad (1.27)$$

hence the osculating circle of γ at s has a parametric representation

$$\vec{y}_{m,s}(t) = \vec{x}_m(s) + \frac{\cosh s}{s}\left(\cos t\vec{v}_1(s) + \sin t\vec{v}_2(s)\right) \text{ for } t \in (0, 2\pi). \quad (1.28)$$

(b) The centre and radius of the osculating sphere of γ at s are

$$\vec{m}(s) = \left\{\frac{1}{2}\cosh s\cos s, \frac{1}{2}\cosh s\sinh s, s\right\} - \frac{1}{2}\sinh s\{-\sin s, \cos s, 0\} \quad (1.29)$$

and

$$r_m(s) = \frac{1}{2}\sqrt{\cosh^2 s + \sinh^2 s}. \quad (1.30)$$

Proof. (a) The radius of curvature of γ at s is by (1.24)

$$\rho(s) = \frac{1}{\kappa(s)} = \frac{\cosh s}{2}.$$

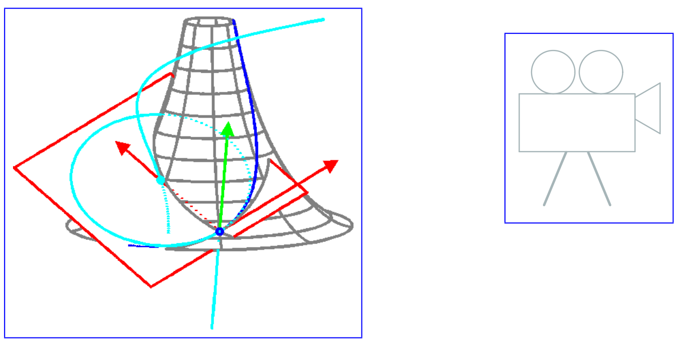
Therefore we obtain from (1.19) and (1.22) for the centre of curvature of γ at s

$$\vec{x}_m(s) = \vec{x}^*(s) + \rho(s)\vec{v}_2(s) =$$
$$\left\{\frac{\cos s}{\cosh s}, \frac{\sin s}{\cosh s}, s - \frac{\sinh s}{\cosh s}\right\} - \left\{\frac{\cos s}{2\cosh s}, \frac{\sin s}{2\cosh s}, -\frac{\sinh s}{2\cosh s}\right\} -$$
$$\frac{\sinh s}{2}\{-\sin s, \cos s, 0\} =$$

$$\left\{\frac{\cos s}{2\cosh s}, \frac{\sin s}{2\cosh s}, s - \frac{1}{2}\tanh s\right\} - \frac{\sinh s}{2}\{-\sin s, \cos s, 0\}$$

which is (1.27). Now (1.28) follows immediately from the definition of the osculating circle of a curve at s.

Figure 9 *Osculating plane and circle*



(b) Since $\dot{\rho}(s) = \sinh(s)/2$ and $\tau(s) = 1$ by (1.25), we obtain the centre of the osculating sphere of γ from (1.23) and (1.27)

$$\vec{m}(s) = \vec{x}^*(s) + \rho(s)\vec{v}_2(s) + \frac{\dot{\rho}(s)}{\tau(s)}\vec{v}_3(s) = \vec{x}_m(s) + \frac{\sinh s}{2}\vec{v}_3(s)$$

$$= \left\{\frac{\cos s}{2\cosh s}, \frac{\sin s}{2\cosh s}, s - \frac{1}{2}\tanh s\right\} - \frac{1}{2}\sinh s\{-\sin s, \cos s\} +$$
$$+ \left\{\frac{\sinh^2 s\cos s}{2\cosh s}, \frac{\sinh^2 s\sin s}{2\cosh s}, \frac{\tanh s}{2s}\right\}$$

$$= \left\{\frac{1}{2}\cosh s\cos s, \frac{1}{2}\cosh s\sinh s, s\right\} - \frac{1}{2}\sinh s\{-\sin s, \cos s, 0\}$$

which is (1.29). Finally, we obtain for the radius of the osculating sphere

$$r_m^2(s) = \rho^2(s) + \left(\frac{\dot{\rho}(s)}{\tau(s)}\right)^2 = \frac{1}{4}\left(\cosh^2 s + \sinh^2 s\right),$$

and (1.30) is an immediate consequence. □

Now we consider the curve of the centres of the osculating spheres of the asymptotic line.

Proposition 1.11 *Let γ be the asymptotic line with a parametric representation (1.19) and γ_m be the curve of the centres of the osculating spheres along γ. Then the arc length s^* along γ_m is given by*

$$s^*(s) = \sinh s, \tag{1.31}$$

and γ_m has the following parametric representation with respect to s^*

$$\vec{m}^*(s^*) =$$
$$\left\{\frac{1}{2}\sqrt{1+(s^*)^2}\cos\left(\psi^*(s^*)\right), \frac{1}{2}\sqrt{1+(s^*)^2}\sin\left(\psi^*(s^*)\right), \psi^*(s^*)\right\} -$$
$$\frac{s^*}{2}\{-\sin\left(\psi^*(s^*)\right), \cos\left(\psi^*(s^*)\right), 0\} \text{ where } \psi^*(s^*) = \operatorname{Arsinh}(s^*). \tag{1.32}$$

The vectors $\vec{v}^*(s^*)$ of the trihedra, the curvature $\kappa^*(s^*)$ of γ_m at s^* are

$$\vec{v}_1^*(s^*) = \left\{\frac{s^*}{\sqrt{1+(s^*)^2}}\cos(\psi^*(s^*)), \frac{s^*}{\sqrt{1+(s^*)^2}}\sin(\psi^*(s^*)), \frac{1}{\sqrt{1+(s^*)^2}}\right\}, \tag{1.33}$$

$$\vec{v}_2^*(s^*) = \frac{1}{\sqrt{1+(s^*)^2}}\left(\frac{1}{\sqrt{1+(s^*)^2}}\{\cos(\psi^*(s^*)), \sin(\psi^*(s^*)), -s^*\} + \{-s^*\sin(\psi^*(s^*)), s^*\cos(\psi^*(s^*)), 0\}\right). \tag{1.34}$$

$$\vec{v}_3^*(s^*) = \frac{1}{\sqrt{1+(s^*)^2}}\{-\sin(\psi^*(s^*)), \cos(\psi^*(s^*)), 0\} - \frac{s^*}{1+(s^*)^2}\{\cos(\psi^*(s^*)), -\sin(\psi^*(s^*)), -s^*\} \tag{1.35}$$

and

$$\kappa^*(s^*) = \frac{1}{\sqrt{1+(s^*)^2}} \tag{1.36}$$

Figure 10 *Osculating plane and sphere*

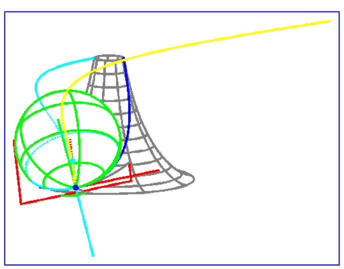
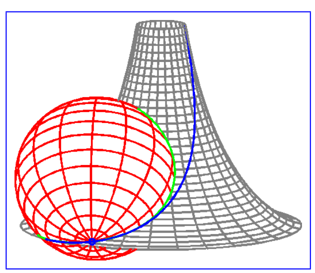
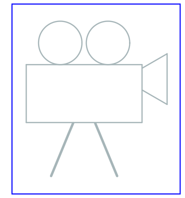


Figure 11 *Osculating sphere*





Proof. Omitting the argument s, and using Frenet's formulae and $\tau = 1$, we obtain

$$\begin{aligned}\frac{d\vec{m}}{ds} &= \frac{d}{ds}\left(\vec{x}^* + \rho\vec{v}_2 + \frac{\dot{\rho}}{\tau}\right) = \vec{v}_1 + \dot{\rho}\vec{v}_2 + \rho\dot{\vec{v}}_2 + \ddot{\rho}\vec{v}_3 + \dot{\rho}\dot{\vec{v}}_2 \\ &= \vec{v}_1 + \dot{\rho}\vec{v}_2 + \rho\left(-\frac{1}{\rho}\vec{v}_1 + \vec{v}_3\right) + \ddot{\rho}\vec{v}_3 - \dot{\rho}\vec{v}_2 = (\rho + \ddot{\rho})\,\vec{v}_3.\end{aligned}$$

Since $\rho(s) = \ddot{\rho}(s) = \cosh s/2$, it follows from (1.24) that

$$\frac{d\vec{m}(s)}{ds} = \cosh s\,\vec{v}_3(s) = \{\sinh s\cos s, \sinh s\sin s, 1\} \tag{1.37}$$

and

$$\left\|\frac{d\vec{m}(s)}{ds}\right\| = \sqrt{\sinh^2 s + 1} = |\cosh s| = \cosh s,$$

hence the arc length s^* along γ_m is given by

$$s^*(s) = \int \left\|\frac{d\vec{m}(s)}{ds}\right\| ds = \int \cosh s\,ds = \sinh s,$$

which is (1.31). This implies

$$s = \operatorname{Arsinh}(s^*) = \log\left(s^* + \sqrt{(s^*)^2+1}\right) = \psi^*(s^*).$$

Substituting this in (1.29) and observing that $\cosh s = \sqrt{1+(s^*)}$, we obtain (1.32).
Since $\vec{m}^*(s^*) = \vec{m}(s(s^*))$, an application of the chain rule and (1.37) and (1.23) together yield

$$\vec{v}_1^*(s^*) = \frac{d\vec{m}^*(s^*)}{ds^*} = \frac{d\vec{m}(s(s^*))}{ds}\frac{ds(s^*)}{ds^*} =$$
$$\cosh(s(s^*))\vec{v}_3(s(s^*))\frac{1}{\cosh(s(s^*))} =$$
$$\left\{\frac{s^*}{\sqrt{1+(s^*)^2}}\cos(\psi^*(s^*)), \frac{s^*}{\sqrt{1+(s^*)^2}}\sin(\psi^*(s^*)), \frac{1}{\sqrt{1+(s^*)^2}}\right\}$$

which is (1.33).
Since $\vec{v}_1(s^*) = \vec{v}_3(s(s^*))$, and application of the chain rule and (1.26) together yield

$$\begin{aligned}\dot{\vec{v}}_1^*(s^*) &= \frac{d\vec{v}_3(s(s^*))}{ds}\frac{ds(s^*)}{ds^*}\\ &= \frac{1}{\cosh^2(s(s^*))}\left(\frac{1}{\cosh(s(s^*))}\{\cos(\psi^*(s^*)), \sin(\psi^*(s^*)), -s^*\}\right.\\ &\qquad \left. + \{-s^*\sin(\psi^*(s^*)), s^*\cos(\psi^*(s^*)), 0\}\right)\\ &= \frac{1}{1+(s^*)^2}\left(\frac{1}{\sqrt{1+(s^*)^2}}\{\cos(\psi^*(s^*)), \sin(\psi^*(s^*)), -s^*\}\right.\\ &\qquad \left. + \{-s^*\sin(\psi^*(s^*)), s^*\cos(\psi^*(s^*)), 0\}\right).\end{aligned}$$

Now we obtain the curvature $\kappa^*(s^*)$ of γ_m at s^* from

$$\begin{aligned}(\kappa^*(s^*))^2 &= \|\dot{\vec{v}}_1^*(s^*)\|^2\\ &= \frac{1}{(1+(s^*)^2)^2}\left(\frac{1}{1+(s^*)^2}(1+s^2)+(s^*)^2\right)\\ &= \frac{1}{(1+(s^*)^2)^2}(1+s^2) = \frac{1}{1+(s^*)^2},\end{aligned}$$

which yields (1.36).
Furthermore, we obtain (1.34) from

$$\vec{v}_2^*(s^*) = \dot{\vec{v}}_1(s^*)\frac{1}{\kappa^*(s^*)}.$$

Since

$$\begin{aligned}\vec{a}(s^*) &= \{s^*\cos(\psi^*(s^*)), s^*\sin(\psi^*(s^*)), 1\}\\ &\qquad \times \{\cos(\psi^*(s^*)), \sin(\psi^*(s^*)), -s^*\}\\ &= \{-(1+(s^*)^2)\sin(\psi^*(s^*)), (1+(s^*)^2)\cos(\psi^*(s^*)), 0\}\\ &= (1+(s^*)^2)\{-\sin(\psi^*(s^*)), \cos(\psi^*(s^*)), 0\},\\ \vec{b}(s^*) &= \{s^*\cos(\psi^*(s^*)), s^*\cos(\psi^*(s^*)), 1\}\\ &\qquad \times \{-\sin(\psi(s^*)), \cos(\psi(s^*)), 0\}\\ &= \{-\cos(\psi^*(s^*)), -\sin(\psi^*(s^*)), s^*\}\end{aligned}$$

and

$$\vec{v}_3^*(s^*) = \frac{1}{\left(\sqrt{1+(s^*)^2}\right)^3}\vec{a} + \frac{s^*}{1+(s^*)^2}\vec{b},$$

we obtain (1.35). □

Figure 12 *Surface generated by the osculating planes*

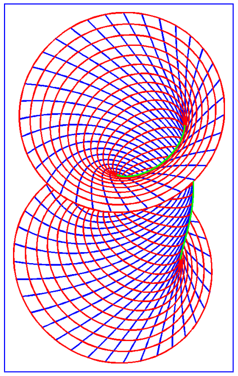
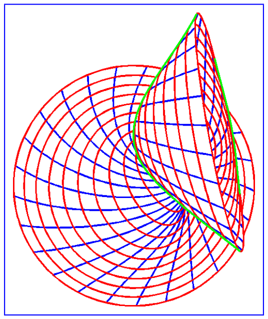


Figure 13 *Envelope of the osculating spheres*



1.6 Geodesic Lines on the Pseudo-Sphere

Here we determine the geodesic lines on the pseudo–sphere with a parametric representation (1.16), which are given by the differential equations

$$\ddot{u}^i + \left\{ \begin{array}{c} i \\ jk \end{array} \right\} \dot{u}^i \dot{u}^k = 0 \text{ for } i = 1, 2, \tag{1.38}$$

where $\left\{ \begin{array}{c} i \\ jk \end{array} \right\}$ $(i, j, k = 1, 2)$ denote the second Christoffel symbols, and summation is carried out in (1.38) with respect to $1 \leq j, k \leq 2$.

Proposition 1.12 *Let $(u_0^1, u_0^2) \in (0, \infty) \times (0, 2\pi)$ be given. Then the u^1–line corresponding to u_0^2 is a geodesic line. Furthermore, if $\Theta_0 \in [0, 2\pi) \setminus \{\pi/2, 3\pi/2\}$ then the geodesic line through (u_0^1, u_0^2) with an angle of Θ_0 to the u_2–line through (u_0^1, u_0^2) is given by*

$$u^1(s) = -\log\left(|\delta| \cosh\left(s + s_0\right)\right) \text{ and } u^2(s) = \frac{1}{\delta} \tanh\left(s + s_0\right) + c_1 \tag{1.39}$$

where

$$\delta = \cos\Theta_0 e^{-u_0^1},\ c_1 = u_0^2 + e^{u_0^1} \tan\Theta_0 \text{ and } s_0 = \log\left(\sqrt{\frac{1 - \sin\Theta_0}{1 + \sin\Theta_0}}\right). \tag{1.40}$$

Figure 14 *A geodesic line*

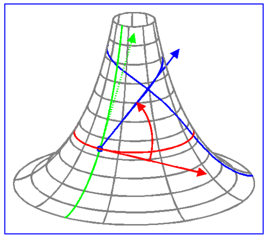
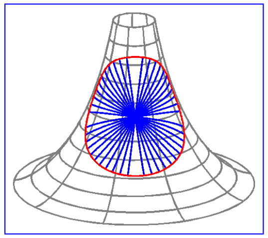


Figure 15 *Families of geodesic lines on a speudo–sphere*



Proof. Since the first fundamental coefficients for the pseudo–sphere are

$$g_{11}(u^i) = g_{11}(u^1) = 1,\ g_{12}(u^i) = 0 \text{ and } g_{22}(u^i) = g_{22}(u^1) = e^{-2u^1},$$

we obtain the second Christoffel symbols

$$\left\{ \begin{array}{c} 1 \\ 11 \end{array} \right\} = \frac{1}{2g_{11}(u^1)} \frac{dg_{11}(u^1)}{du^1} = 0,$$
$$\left\{ \begin{array}{c} 1 \\ 12 \end{array} \right\} = \left\{ \begin{array}{c} 1 \\ 21 \end{array} \right\} = 0,$$
$$\left\{ \begin{array}{c} 1 \\ 22 \end{array} \right\} = -\frac{1}{2g_{11}(u^1)} \frac{g_{22}(u^1)}{du^1} = e^{-2u^1},$$
$$\left\{ \begin{array}{c} 2 \\ 11 \end{array} \right\} = 0,$$
$$\left\{ \begin{array}{c} 2 \\ 12 \end{array} \right\} = \left\{ \begin{array}{c} 2 \\ 21 \end{array} \right\} = \frac{1}{2g_{22}(u^1)} \frac{dg_{22}(u^1)}{du^1} = -1$$

and

$$\left\{ \begin{array}{c} 2 \\ 22 \end{array} \right\} = 0,$$

and the differential equations (1.38) reduce to

$$\ddot{u}^1 + e^{-2u^1} \left(\dot{u}^2\right)^2 = 0 \tag{1.41}$$

and

$$\ddot{u}^2 - 2\dot{u}^1 \dot{u}^2 = 0. \tag{1.42}$$

If $\dot{u}^2 = 0$ then we obtain a u^1–line corresponding to $u^2 = u_2^0$, that is, for a suitable orientation of the arc length

$$u^1 = s + u_0^1 \text{ and } u_2(s) = u_0^2 \text{ for } s > -u_0^1.$$

Now we assume $\dot{u}^2 \neq 0$. Then (1.42) yields

$$\frac{\ddot{u}^2}{\dot{u}^2} = 2\dot{u}^1, \text{ that is } \dot{u}^2 = \delta e^{2u^1} \text{ for some constant } \delta \neq 0. \tag{1.43}$$

Substituting this in (1.41), we obtain

$$\ddot{u}^1 + \delta^2 e^{2u^1} = 0, \text{ that is } \frac{d}{ds}\left(\left(\dot{u}^1\right)^2 + \delta^2 e^{2u^1}\right) = 0,$$

hence

$$\left(\dot{u}^1\right)^2 = d^2 - \delta^2 e^{2u^1} \text{ for some constant } d \neq 0 \text{ and } u^1 < \log\left|\frac{d}{\delta}\right|.$$

This yields

$$\dot{u}^1 = \pm\sqrt{d^2 - \delta^2 e^{2u^1}} \text{ for } u^1 \leq \log\left|\frac{d}{\delta}\right|. \tag{1.44}$$

and

$$I = \int \frac{du^1}{\sqrt{d^2 - \delta^2 e^{2u^1}}} = \pm(s + s_0) \text{ for some constant } s_0.$$

To evaluate the integral $I(u^1)$, we substitute $x = |\delta| e^{u^1}$ and obtain

$$\begin{aligned} I &= \int \frac{dx}{x\sqrt{d^2 - x^2}} \\ &= -\frac{1}{d}\log\left(\frac{d + \sqrt{d^2 - x^2}}{x}\right) - \frac{1}{d}\log\left(\frac{d + \sqrt{d^2 - \delta^2 e^{2u^1}}}{|\delta| e^{u^1}}\right), \end{aligned}$$

hence

$$\left(|\delta| e^{u^1} e^{(\mp d(s+s_0))} - d\right)^2 = d^2 - \delta^2 e^{2u^1}$$

and

$$e^{u^1} = \frac{2d e^{(\mp d(s+s_0))}}{|\delta|\left(1 + e^{(\mp d(s+s_0))}\right)} = \frac{d}{|\delta| \cosh\left(d(s+s_0)\right)}. \tag{1.45}$$

Therefore we have

$$u^1(s) = \log\left(\frac{d}{|\delta| \cosh\left(d(s+s_0)\right)}\right) \text{ for all } s \in \mathbb{R},$$

since $\cosh\left(d(s+s_0)\right) \geq 1$ for all s implies $u^1(s) \leq \log|d/\delta|$ for all s. Now it follows from the identity on the right hand side in (1.43) and (1.45) that

$$\dot{u}^2 = \frac{d^2}{\delta \cosh^2\left(d(s+s_0)\right)},$$

and so

$$u^2 = \frac{d}{\delta}\tanh\left(d(s+s_0)\right) + c_1 \text{ for some constant } c_1.$$

Furthermore, since s is the arc length along the geodesic line, it follows from (1.44) and the identity on the right hand side in (1.43) that

$$g_{11}(u^1)(\dot{u}^1)^2 + g_{22}(u^1)(\dot{u}^2)^2 = d^2 - \delta^2 e^{2u^1} + \delta^2 e^{2u^1} = d^2 = 1,$$

hence $d = \pm 1$. Therefore, the general solution of the differential equations (1.41) and (1.42) is

$$u^1(s) = -\log|\delta| \cosh\left(s + s_0\right)$$

and

$$u^2(s) = \pm\frac{1}{\delta}\tanh\left(\pm(s+s_0)\right) + c_1 = \frac{1}{\delta}\tanh\left(s+s_0\right) + c_1.$$

Thus we have shown the identities in (1.39).
Now we determine the constants δ, s_0 and c_1 such that the initial conditions are satisfied. Let $\vec{x}^*(s)$ be a parametric representation of the geodesic line. Then it follows from

$$\frac{\dot{\vec{x}}^*(0) \bullet \vec{x}_1(u_0^1)}{\|\vec{x}_1(u_0^1)\|} = \sqrt{g_{11}(u_0^1)}\,\frac{du^1(0)}{ds} = -\tanh s_0 = \sin\Theta_0,$$

that

$$s_0 = \operatorname{Artanh}(-\sin\Theta_0) = \frac{1}{2}\log\left(\frac{1-\sin\Theta_0}{1+\sin\Theta_0}\right),$$

which is the third identity in (1.40). Furthermore,

$$\frac{\dot{\vec{x}}^*(0)\bullet\vec{x}_2(u_0^1)}{\|\vec{x}_2(u_0^1)\|} = \sqrt{g_{22}(u_0^1)}\,\frac{du^2(0)}{ds} = \mathrm{e}^{-u_0^1}\delta\mathrm{e}^{2u_0^1} = \cos\Theta_0$$

implies $\delta = \mathrm{e}^{-u_0^1}\cos\Theta_0$, which is the first identity in (1.40). Finally

$$u_0^2 = \frac{1}{\delta}\tanh s_0 + c_1$$

yields

$$c_1 = u_0^2 + \sin\Theta_0\frac{\mathrm{e}^{u_0^1}}{\cos\Theta_0} = u_0 + \mathrm{e}^{u_0^1}\tan\Theta_0,$$

which is the second identity in (1.40). □

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